Student Name

Mathematics Stage 5 Paths

CIRCLE GEOMETRY

Book 2 Prove and apply tangent and secant properties of circles

Version: 260207

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CONTENTS

OVERVIEW

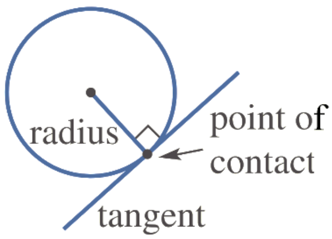
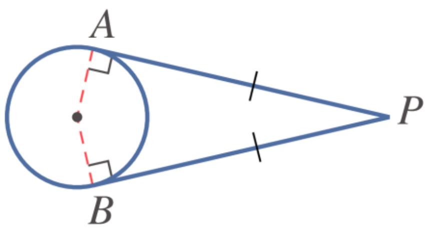
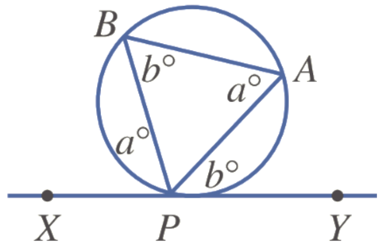
SYLLABUS CONTENT

MA5-CIR-P-01 applies deductive reasoning to prove circle theorems and solve related problems

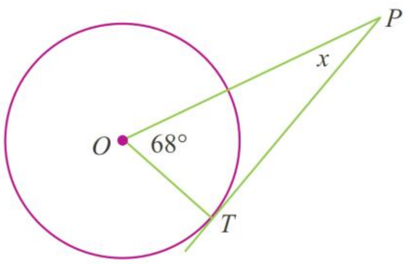
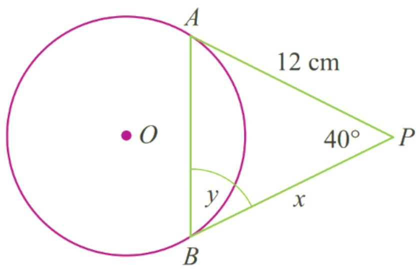
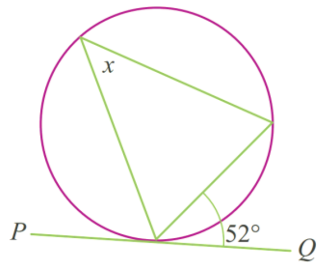
Prove and apply tangent and secant properties of circles

- Prove tangent and secant properties of circles
- Apply tangent and secant properties of circles to find unknown angles and lengths in diagrams

TANGENT AND SECANT PROPERTIES

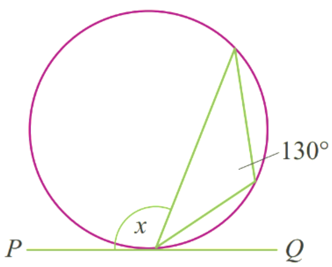
Angle between tangent and radius = 90° 	Tangents from an external point are equal in length. 	**Alternate segment theorem** The angle between a tangent and a chord at the point of contact is equal to the angle in the alternate segment. 

Example Apply tangent and secant properties to find unknown values.

 $\angle OTP = 90°$ (A tangent is perpendicular to the radius at the point of contact) $x + 90° + 68° = 180°$ $x = 22°$	 $x = 12$ cm (Tangents drawn to a circle from an external point are equal.) $y = \dfrac{180 - 40}{2} = 70°$ (Base angles of isosceles triangle equal)	 $x = 52°$ (Alternate segment theorem)

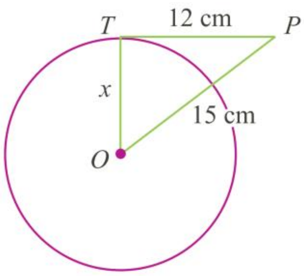
Find the value of the missing value.

a



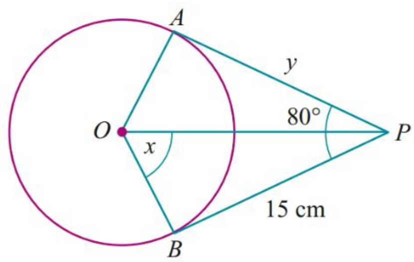
130°

b



9 cm

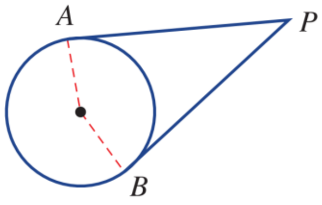
c



$x = 50°, y = 15°$

1

a How many times does a tangent intersect a circle?



once

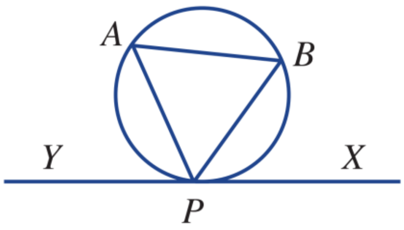
b At the point of contact, what angle does the tangent make with the radius?

90°

c If AP is 5 cm, what is the length of BP in this diagram?

Also 5 cm

2 For this diagram name the angle that is:



a equal to ∠BPX

∠BAP

b equal to ∠BAP

∠BPX

c equal to ∠APY

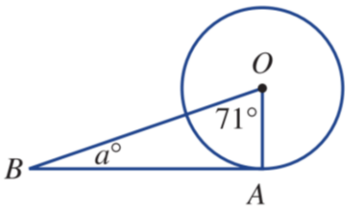
∠ABP

d equal to ∠ABP

∠APY

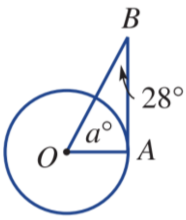
3 Find the value of a in these diagrams that include tangents.

a



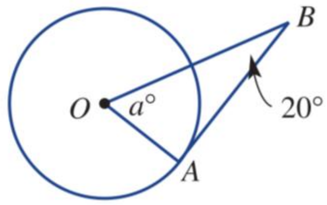
19

b



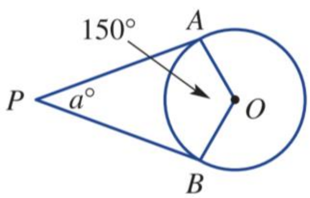
62

c



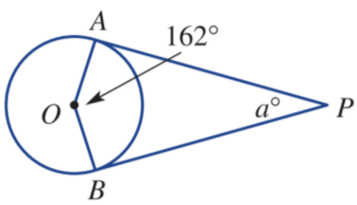
70

d



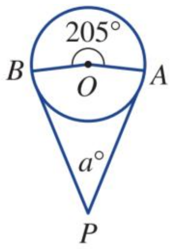
30

e



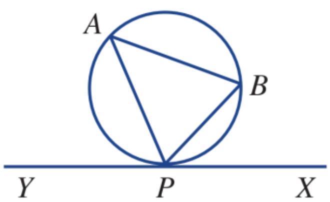
18

f



25

4 In this diagram, XY is a tangent to the circle. Find:



a ∠PAB if ∠BPX = 50

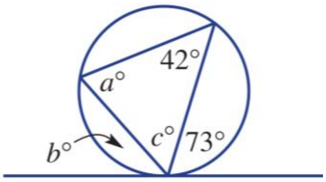
50°

b ∠APY if ∠ABP = 59°

59°

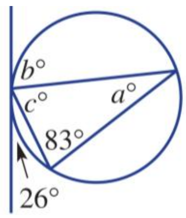
5 Find the value of a, b and c in these diagrams.

a



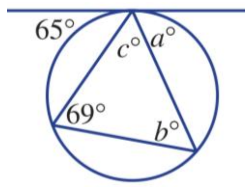
a=73, b=42, c=65

b



a=26, b=83, c=71

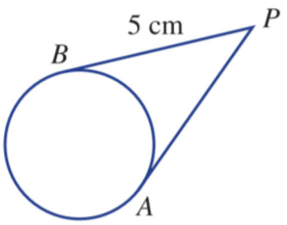
c



a=69, b=65, c=46

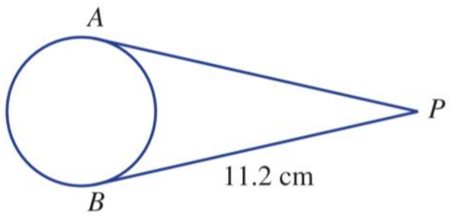
6 Find the length of AP

a



5 cm

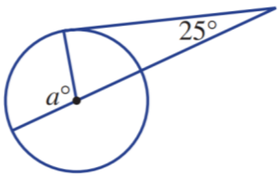
b



11.2 cm

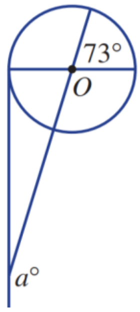
7 Find the value of a

a



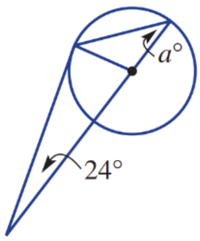
115

b



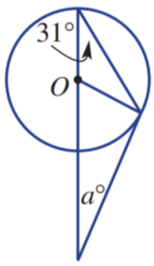
163

c



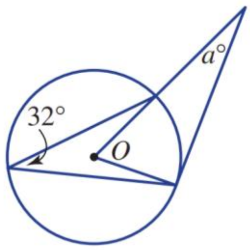
33

d



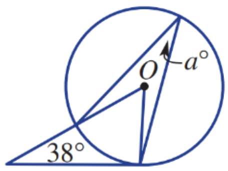
28

e



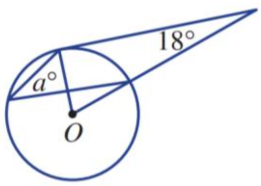
26

f



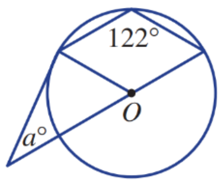
26

g



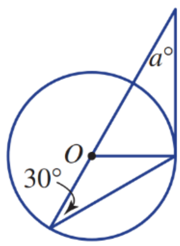
36

h



26

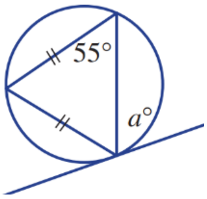
i



30

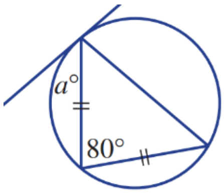
8 Find the value of a.

a



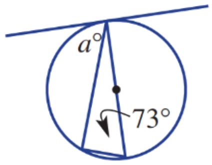
70

b



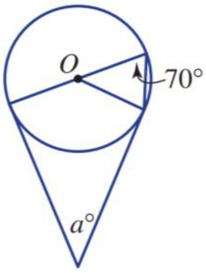
50

c



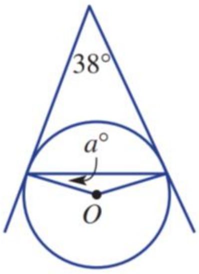
73

d



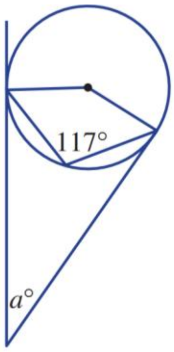
40

e



19

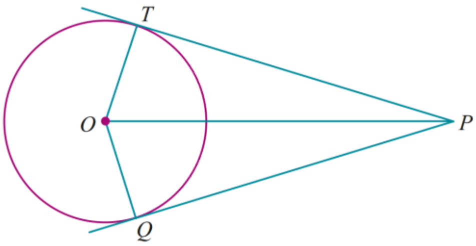
f



54

9 PT and PQ are the tangents from P to a circle with centre O.

a Prove $\triangle PTO \equiv \triangle PQO$.



$\angle OTP = \angle OQT = 90°$ (tangents meet radii at 90°)

$OT = OQ$ (radii)

OP is common

$\triangle PTO \equiv \triangle PQO$ (RHS)

b Hence prove PT = PQ.

$PT = PQ$ (corresponding sides of congruent triangles equal)

Two tangents drawn to a circle from an external point are equal in length.

10 *Proof:* Join CO and BO. Let $\angle BCQ = x$.

$\angle OCB = 90° - x$ (A tangent to a circle is perpendicular to the radius at the point of contact.)

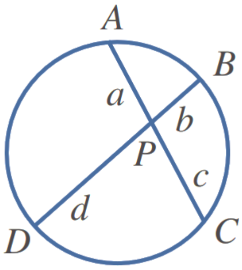
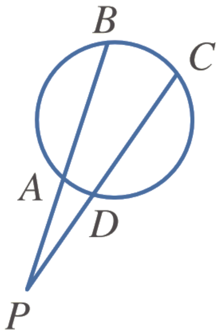
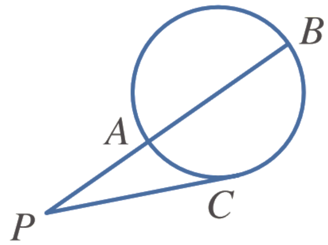
$\angle OBC = 90° - x$ (Base angles of an isosceles $\triangle$ are equal)

$\angle COB = 180° - 2x$ (Angle sum of a $\triangle$ is 180°.)

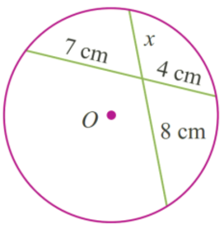
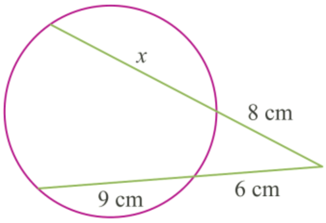
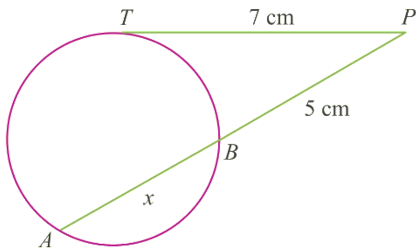
$\angle BAC = 90° - x$ (The angle at the centre of a circle is double the angle at the circumference, standing on the same arc.)

$\therefore \angle BCQ = \angle CAB$

FURTHER TANGENT AND SECANT PROPERTIES

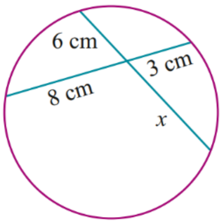
The products of the intercepts of two intersecting chords are equal.	The products of the intercepts of 2 intersecting secants to a circle from an external point are equal.	The square of a tangent to a circle from an external point equals the product of the intercepts of any secants from the point.
		
$ac = bd$	$AP \times BP = DP \times CP$	$CP^2 = AP \times BP$

Example Apply further properties of tangents and secants.

		
$7 \times 4 = 8x$ $28 = 8x$ $\frac{7}{2} = x$	$8(x+8) = 6(6+9)$ $8x + 64 = 90$ $8x = 26$ $x = 3.25$	$7^2 = 5(5+x)$ $49 = 25 + 5x$ $24 = 5x$ $4.8 = x$

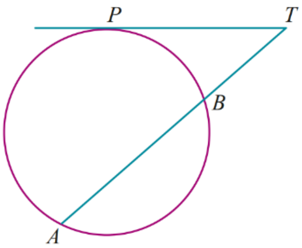
Find the unknown value.

a



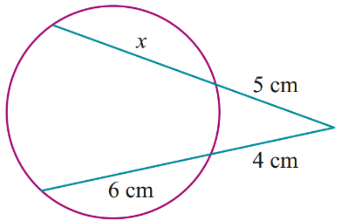
4 cm

b $TA = 8$ cm, $TB = 2$ cm
Find TP



4 cm

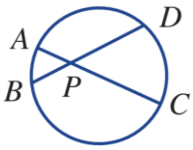
c



3 cm

1 Complete the rules for each diagram.

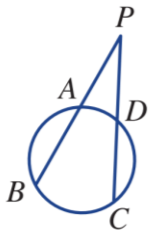
a



$AP \times CP =$ ___ $\times$ ___

AP×CP=BP×DP

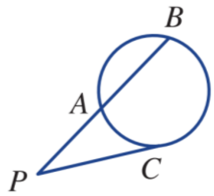
b



$AP \times$ ___ $=$ ___ $\times$ ___

AP×BP=DP×CP

c

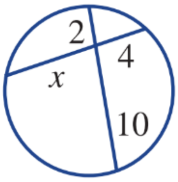


$AP \times$ ___ $=$ ___^{2}

AP×BP=CP²

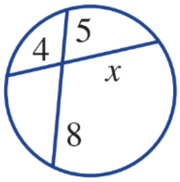
2 Find the length of x in each figure.

a



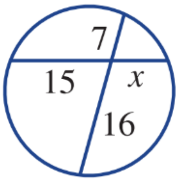
5

b



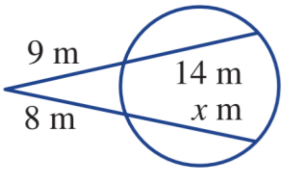
10

c



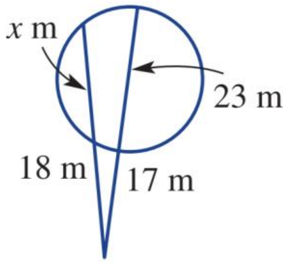
$\frac{112}{15}$

d



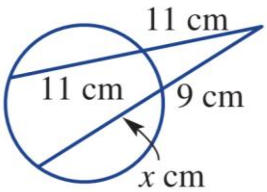
$\frac{143}{8}$

e



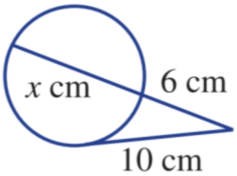
$\frac{178}{9}$

f



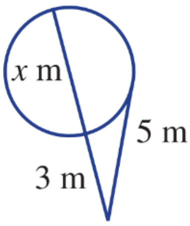
$\frac{161}{9}$

g



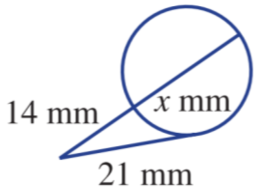
$\frac{32}{3}$

h



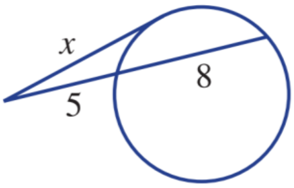
$\frac{16}{3}$

i



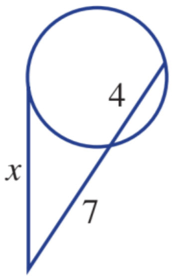
$\frac{35}{2}$

j



$\sqrt{65}$

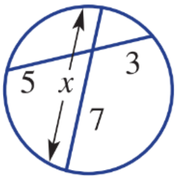
k



$\sqrt{77}$

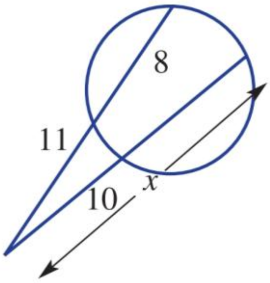
3 Find the value of x.

a



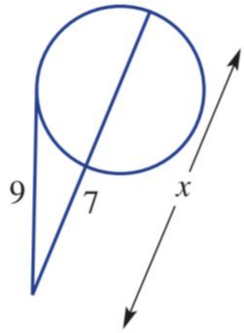
$\frac{64}{7}$

b



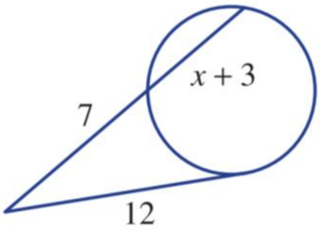
$\frac{209}{10}$

c



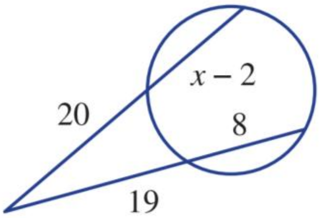
$\frac{81}{7}$

d



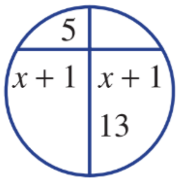
$\frac{74}{7}$

e



$\frac{153}{20}$

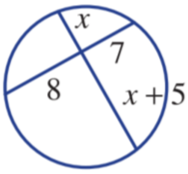
f



$\sqrt{65} - 1$

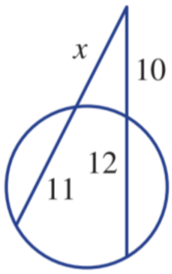
4 For each diagram, derive the equation.

a $x^2 + 5x - 56 = 0$



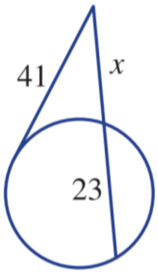
$x(x+5) = 7 \times 8,$
$x^2 + 5x = 56,$
$x^2 + 5x - 56 = 0$

b $x^2 + 11x - 220 = 0$



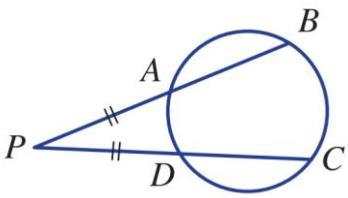
$x(x+11) = 10 \times 22,$
$x^2 + 11x = 220,$
$x^2 + 11x - 220 = 0$

c $x^2 + 23x - 1681 = 0$



$x(x+23) = 41^2,$
$x^2 + 23x = 1681,$
$x^2 + 23x - 1681 = 0$

5 In this diagram AP = DP. Explain why AB = DC.

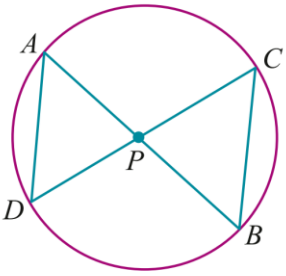


AP² =DP×CP and BP² =DP×CP,

so BP=AP.

6 AB and CD are two chords of a circle that intersect at P.

a Prove that △APD is similar to △CPB



$\angle DAB = \angle DCB$ (angles subtended at circumference by chord DB equal)

$\angle ADC = \angle ABC$ (angles subtended at circumference by chord AC equal)

$\angle ADP = \angle BCP$ (vertically opposite)

$\therefore \triangle APD \,|||\, \triangle CPB$ (equiangular)

b Write down all the equal ratios of sides in these two triangles.

$$\frac{AP}{PC} = \frac{DP}{PB} = \frac{AD}{CB}$$

c Hence show that AP × PB = DP × PC

Cross multiply first two fractions from above.

7

Proof: In $\triangle PTB$ and $\triangle ATP$:

$\angle TPB = \angle PAB$ (The angle between a tangent and a chord is equal to the angle in the alternate segment.)

$\angle PTB = \angle ATP$ (common)

$\triangle PTB$ is similar to $\triangle ATP$. (Two angles of one △ equal two angles of the other △.)

$$\frac{AP}{PB} = \frac{TP}{TB} = \frac{TA}{TP}$$

$\therefore\ TP^2 = TA \times TB.$

8 *Proof:* $PT^2 = TB \times TA$ and $PT^2 = TD \times TC$

(The square of the length of a tangent from an external point equals the product of the intercepts of any secant from the point.)

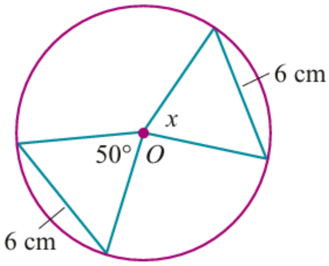
$\therefore TB \times TA = TD \times TC$

MIXED PRACTICE

FOUNDATION

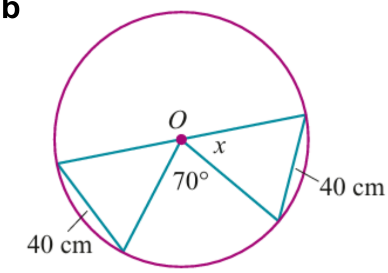
1 Find the value of the pronumeral. You do not need to give reasoning.

a



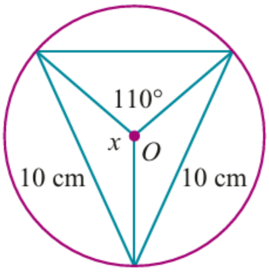
50°

b



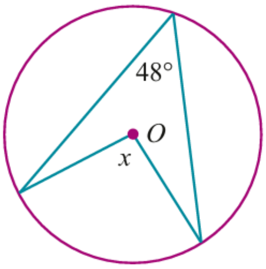
55°

c



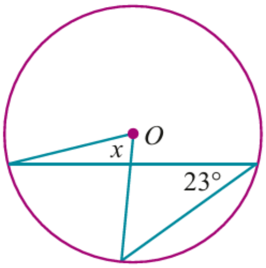
125°

d



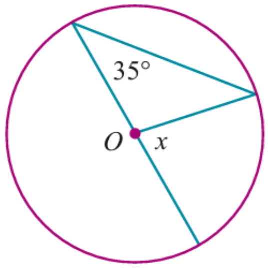
96°

e



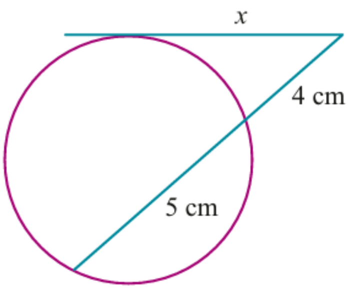
46°

f



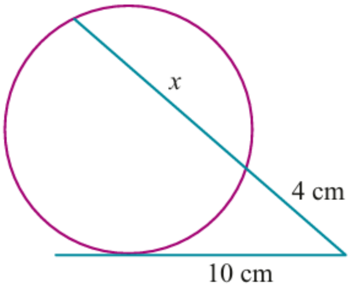
70°

g



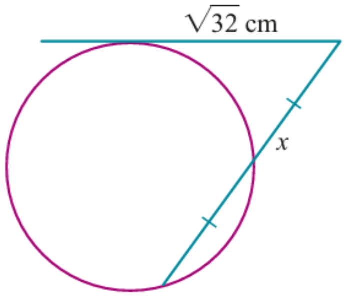
6 cm

h



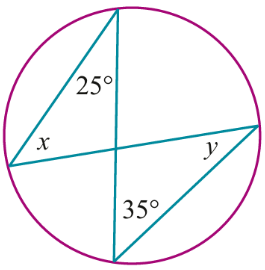
21 cm

i



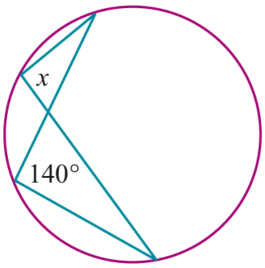
4 cm

j



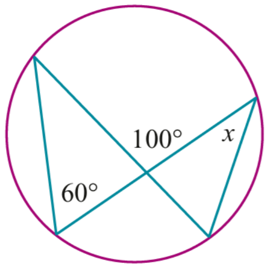
$x = 35°, y = 25°$

k



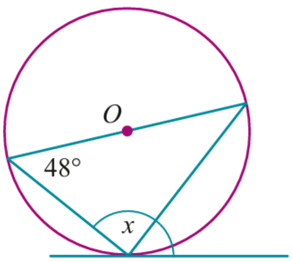
140°

l



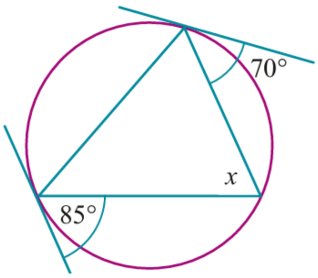
40°

m



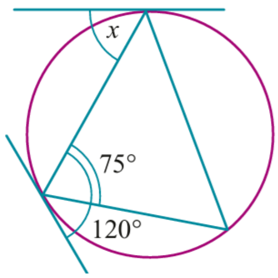
138°

n



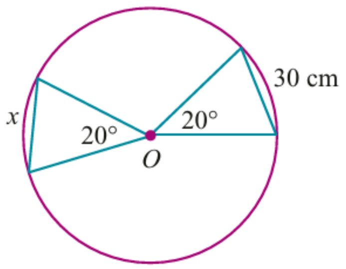
25°

o



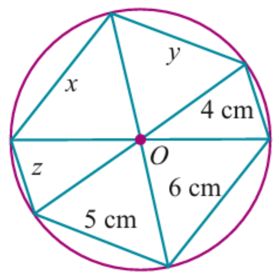
60°

p



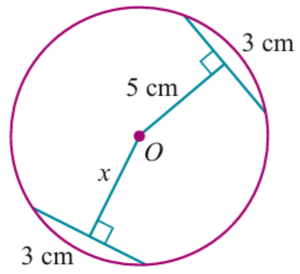
30 cm

q



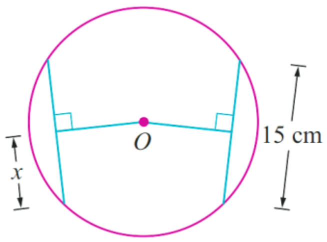
x = 6 cm y = 5 cm z = 4 cm

r



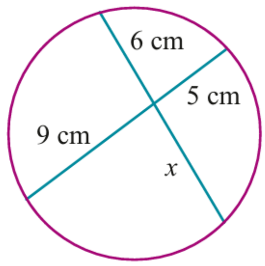
5 cm

s



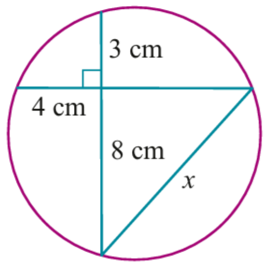
7.5 cm

t



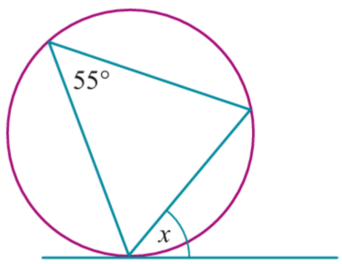
7.5 cm

u



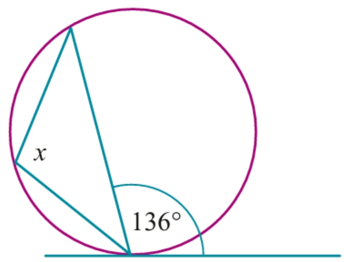
10 cm

v



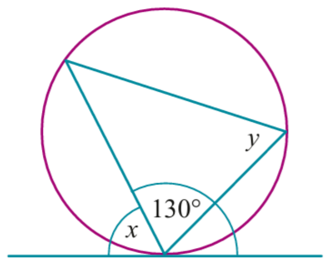
55°

w



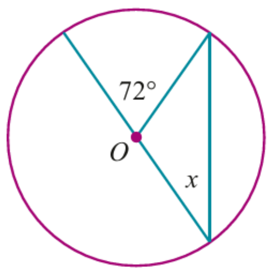
136°

x



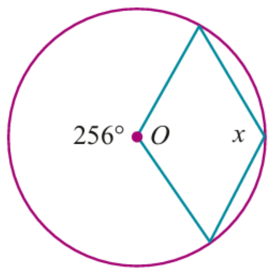
$x = 50°, y = 50°$

y



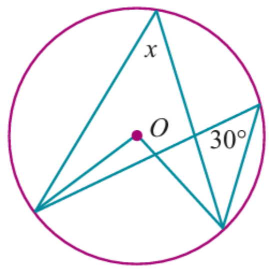
36°

z



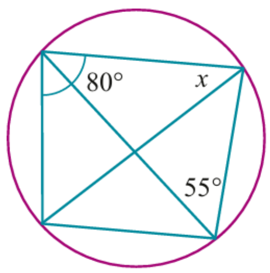
128°

aa



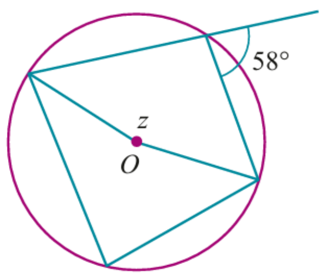
30°

bb



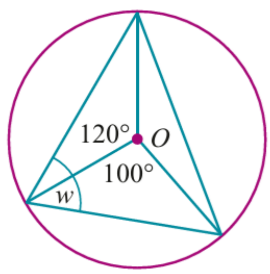
45°

cc



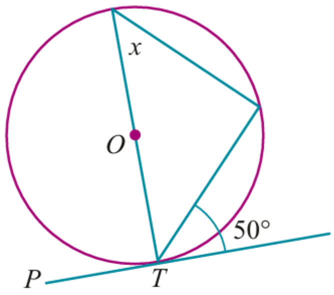
116°

dd



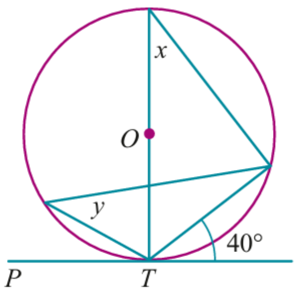
70°

ee



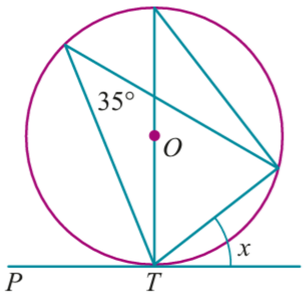
50°

ff



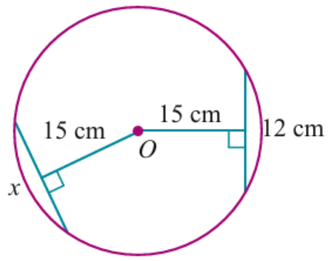
$x = 40°, y = 40°$

gg



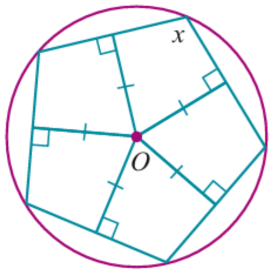
35°

hh



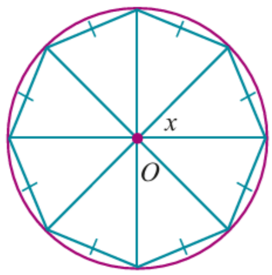
12 cm

ii



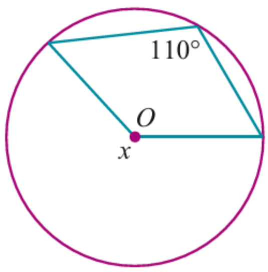
108°

jj



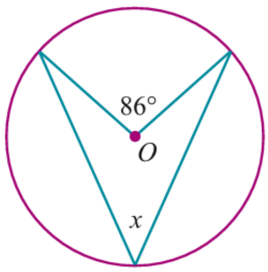
45°

kk



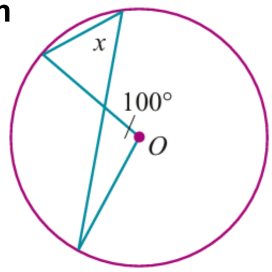
220°

ll



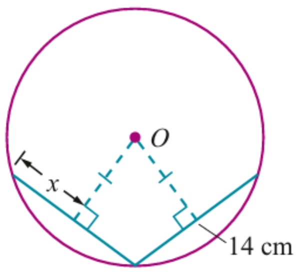
43°

mm



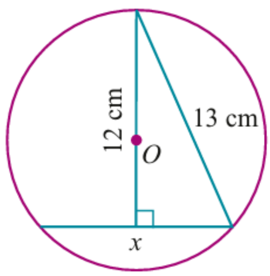
50°

nn



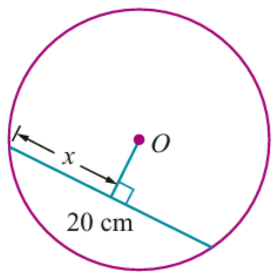
7 cm

oo



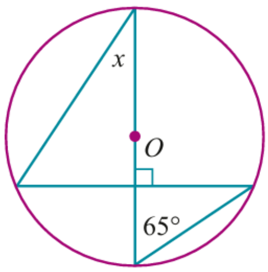
10 cm

pp



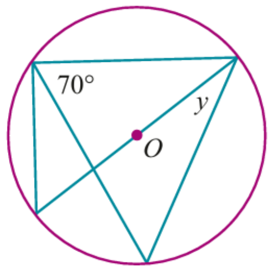
10 cm

qq



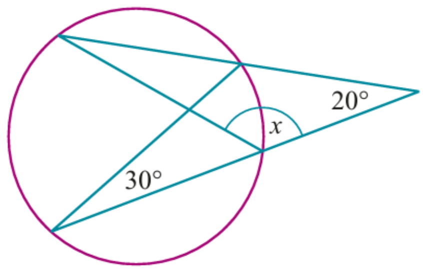
25°

rr



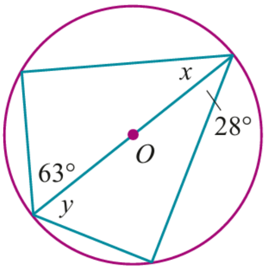
20°

ss



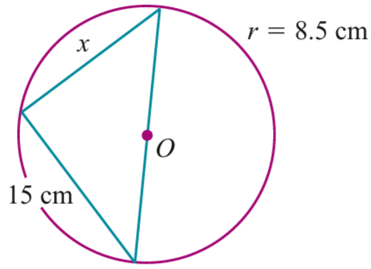
130°

tt



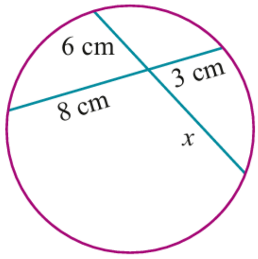
$x = 27°, y = 62°$

uu



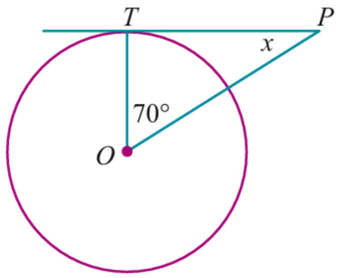
8 cm

vv



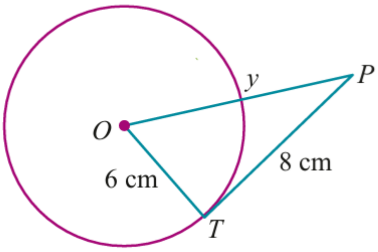
4 cm

ww



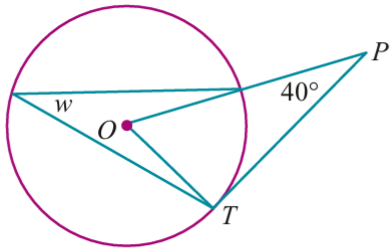
20°

xx



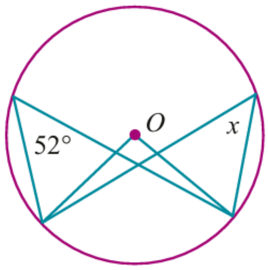
10 cm

yy



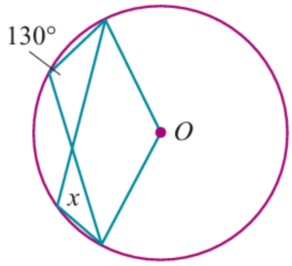
25°

zz



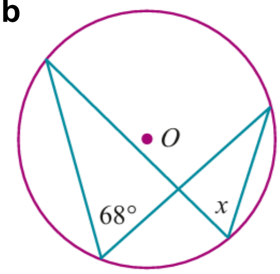
52°

aaa



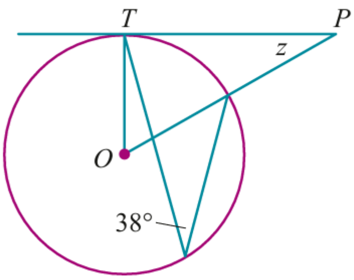
130°

bbb



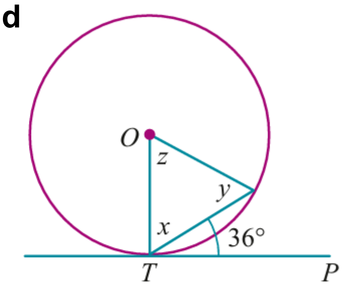
68°

ccc



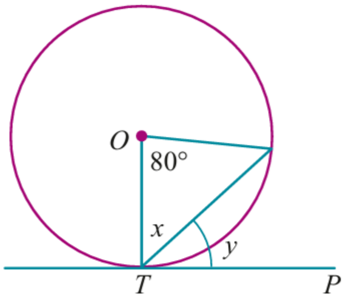
14°

ddd



$x = 54°, y = 54°, z = 72°$

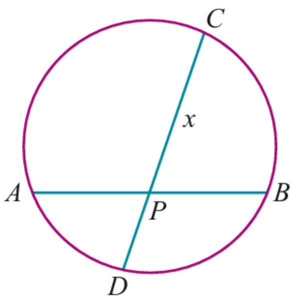
eee



$x = 50°, y = 40°$

fff $AB = 40$ cm, $PD = 25$ cm

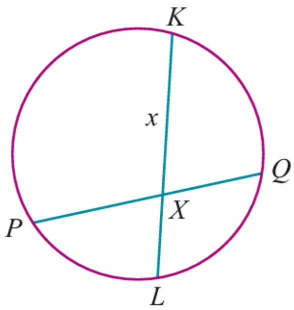
CD bisects AB



16 cm

ggg $PQ = 24$ cm, $XQ = 8$ cm

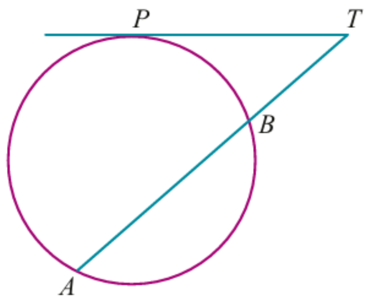
$XL = 10$ cm



12.8 cm

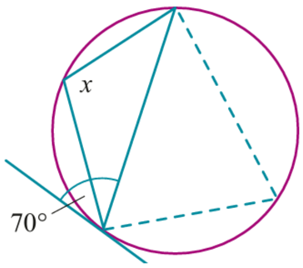
hhh $TA = 8$ cm, $TB = 2$ cm

Find TP



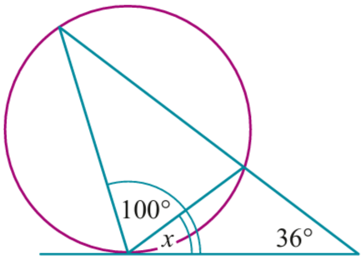
4 cm

iii



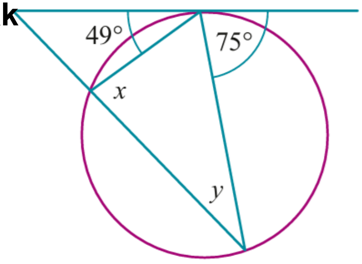
110°

jjj



44°

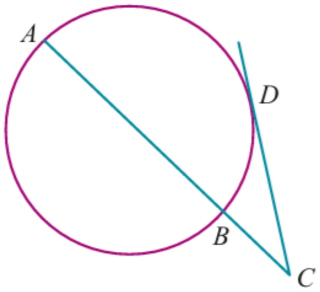
kkk



$x = 75°, y = 49°$

lll $CA = 12$ cm, $CB = 3$ cm

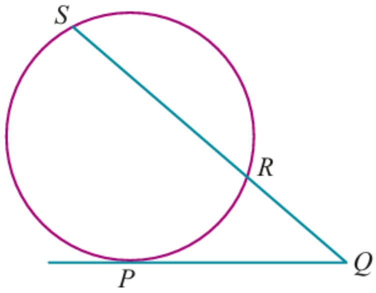
Find CD



6 cm

mmm $QP = 10$ cm, $QR =$ 5 cm

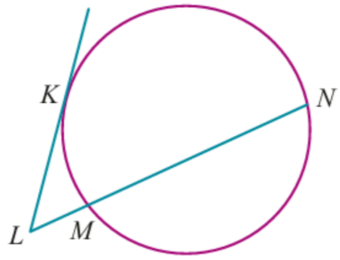
Find QS and RS



QS = 20 cm, RS = 15 cm

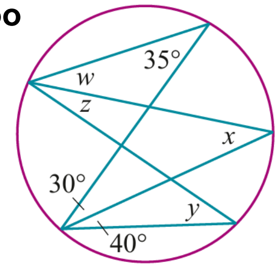
nnn $LK = 12$ cm, $LN = 16$ cm

Find LM and MN



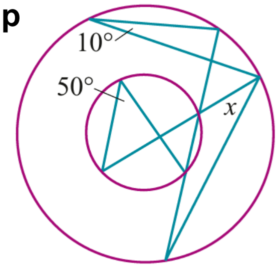
LM = 9 cm, MN = 7 cm

ooo



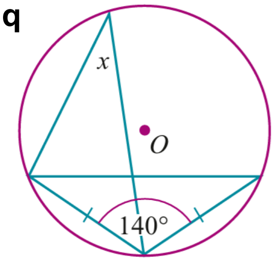
$w = 30°,\ x = 35°, y = 35°, z = 40°$

ppp



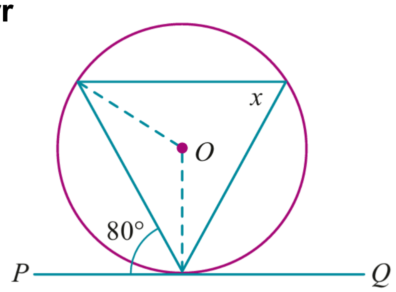
40°

qqq



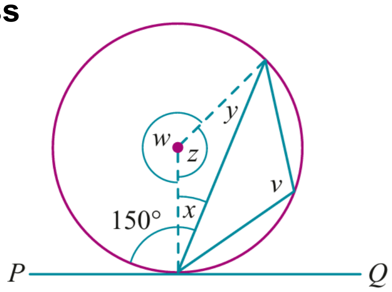
20°

rrr



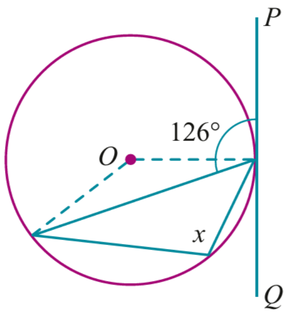
80°

sss



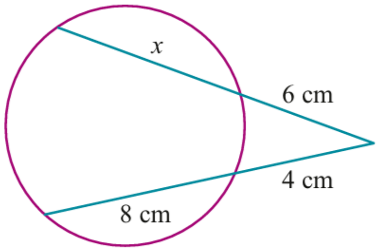
$x = 60°, y = 60°, z = 60°,$
$w = 300°, v = 150°$

ttt



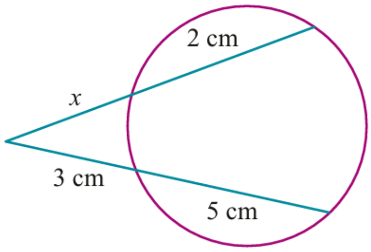
126°

uuu



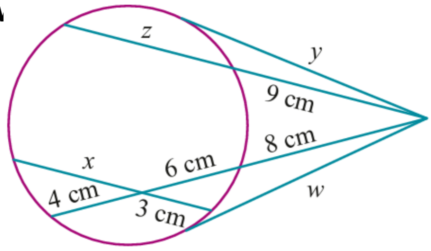
2 cm

vvv



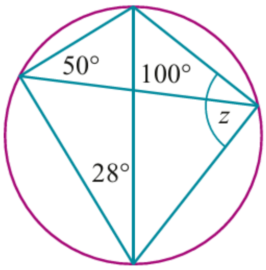
4 cm

www



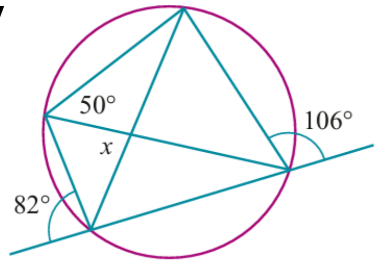
w = 12 cm, x = 8 cm
y = 12 cm, z = 7 cm

xxx



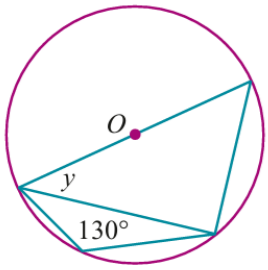
78°

yyy



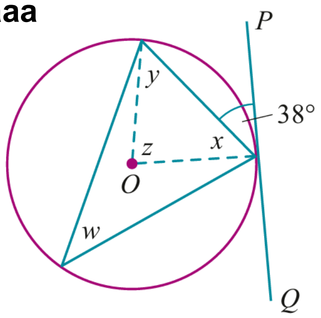
76°

zzz



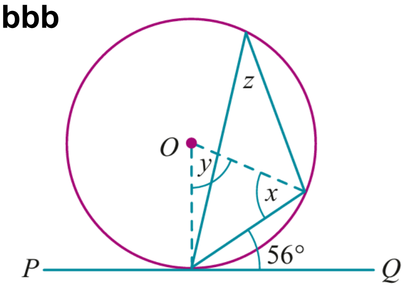
40°

aaaa



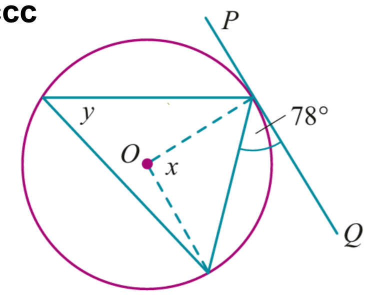
$x = 52°, y = 52°, z = 76°, w = 38°$

bbbb



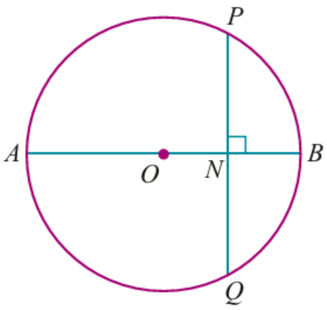
$x = 34°, y = 112°, z = 56°$

cccc



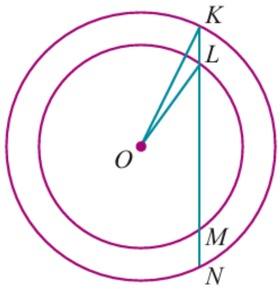
$x = 156°, y = 78°$

2 AN = 18 cm and NB = 8 cm. Find the length of PQ.



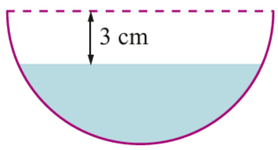
24 cm

3 The figure shows two concentric circles, centre O. OL = 13 cm, OK = 15 cm and LM = 10 cm. Find KN.



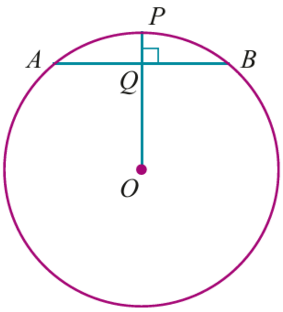
18 cm

4 A hemispherical bowl has a diameter of 10 cm.
Water is poured into the bowl until its surface is 3 cm below the top of the bowl as shown.
Find the diameter of the water surface.



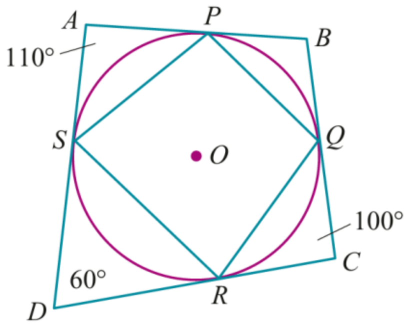
8 cm

5 OP is perpendicular to the chord AB. If AB = 14 cm and PQ = 1 cm, find the radius of the circle.



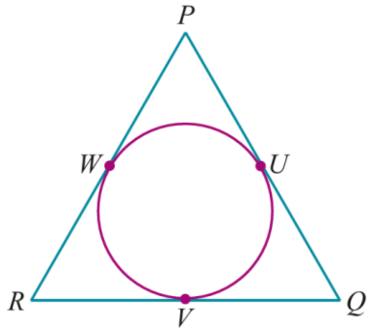
25 cm

6 The circle touches the sides of the quadrilateral ABCD at P, Q, R and S as shown. Find the angles of the quadrilateral PQRS.



$\angle P = 100°, \angle Q = 95°, \angle R = 80°, \angle S = 85°$

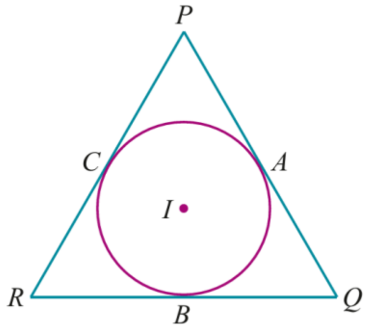
7 PQ, QR and RP are tangents. If PR = 9 cm, RW = 5 cm and RQ = 12 cm, find PQ.



$PQ = 11$ cm

8 I is the incentre of △PQR. RQ = 8 cm, PR = 9 cm and PQ = 7 cm. Find RB.

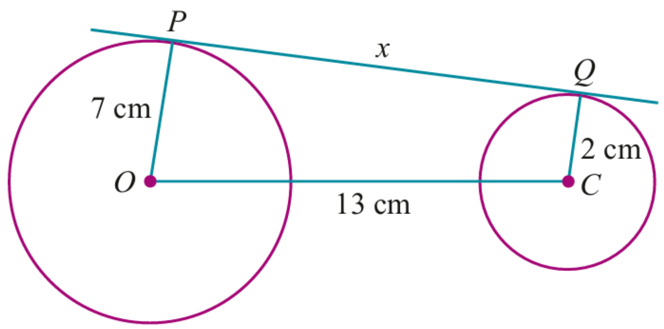
Hint: set up a system of 3 simultaneous equations.



$RB = 5$ cm

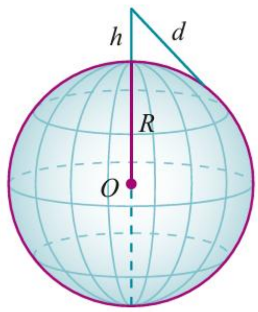
9 Two circles of radii 2 cm and 7 cm have their centres 13 cm apart.

Find the length of the direct common tangent PQ.



12 cm

10 A person of height h metres stands on the surface of Earth, which is assumed to be a sphere of radius R.



a Show that the distance, d, to the horizon is given by $d = \sqrt{h^2 + 2hR}$

$d^2 = h(h + 2R)$

The square of a tangent to a circle from an external point is the product of the intercepts of any secants from the point.

b Calculate the distance across level ground to the horizon that can be seen by a person of height 1.8 m. Use $R = 6400$ km.

4800.000338 m

c The formula in part **a** is often given as $d \approx \sqrt{2hr}$. Why might this be the case?

h^2 is extremely small compared to $2hR$.

EXAM QUESTIONS

1 2019 HSC Extension 1 Band 3

From the point P outside a circle, a secant and a tangent to the circle are constructed a shown in the diagram:

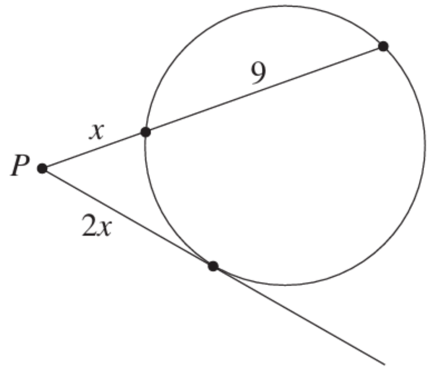
Which equation is satisfied by x?

A. $4x^2 = 9x$

B. $4x^2 = 9 + x$

C. $4x^2 = 9(9 + x)$

D. $4x^2 = x(9 + x)$



D

2 2017 HSC Extension 1 Band 3

The points A, B, C and D lie on a circle and the tangents at A and B meet at T.

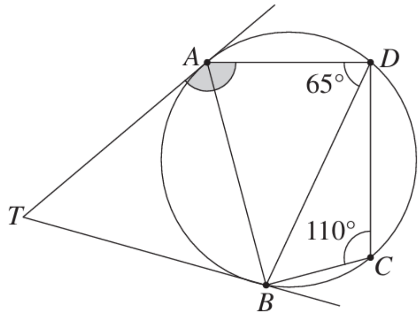
What is the value of $\angle TAD$?

A. 130°

B. 135°

C. 155°

D. 175°

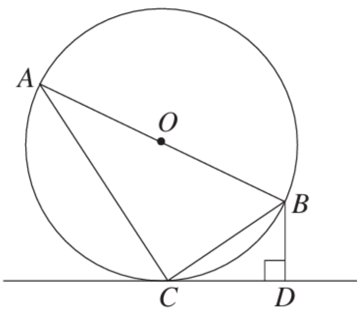


B

3 2019 HSC Extension 1 Band 4

The line segment AB is a diameter of the circle, centred at O. The line CD is the tangent to the circle at C and BD is perpendicular to CD, as shown in the diagram.

Prove that ∠ABC = ∠CBD



Let $\angle BCD = \alpha$

$\therefore$ $\angle CAB = \alpha$ (angle in alt segment)

$\angle ACB = 90^\circ$ (angle in a semi-circle)

$\therefore$ $\angle ABC = 90 - \alpha$ (angle sum $\triangle$)

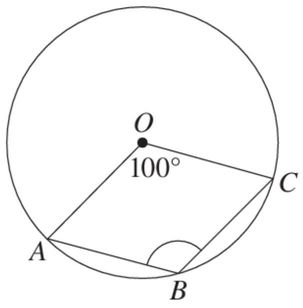
$\angle CBD = 90 - \alpha$ (angle sum $\triangle$)

$\therefore$ $\angle ABC = \angle CBD$

4 2017 HSC Extension 1 Band 4

The points A, B and C lie on a circle with centre O, as shown in the diagram.

The size of $\angle AOC$ is 100°. Find the size of $\angle ABC$, giving reasons.



Reflex $\angle AOC = 360° - 100°$ (angle of revolution)

$= 260°$

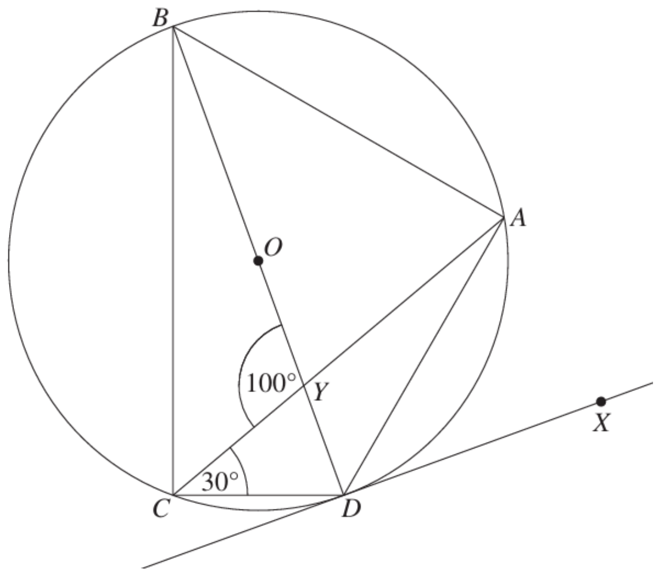
$\angle ABC = \frac{1}{2} \times \text{reflex } \angle AOC$ (angle at centre equals twice the angle at circumference standing in the same arc)

$= \frac{260}{2}$

$= 130°$

5 2015 HSC Extension 1 Band 4

aaaIn the diagram, the points A, B, C and D are on the circumference of a circle, whose centre O lies on BD. The chord AC intersects the diameter BD at Y. The tangent at D passes through the point X. It is given that $\angle CYB = 100°$ and $\angle DCY = 30°$



a What is the size of ∠ACB?

∠ACB + ∠ACD = 90° (Angle in a semi-circle is a right angle)

∠ACB + 30° = 90°

∴ ∠ACB = 60°

b What is the size of ∠ADX?

∠ADX = ∠ACD (alternate segment theorem) = 30°

c Find, giving reasons, the size of ∠CAB.

∠CDY + ∠YCD = 100° (∠BYC, exterior ∠ of △CDY)

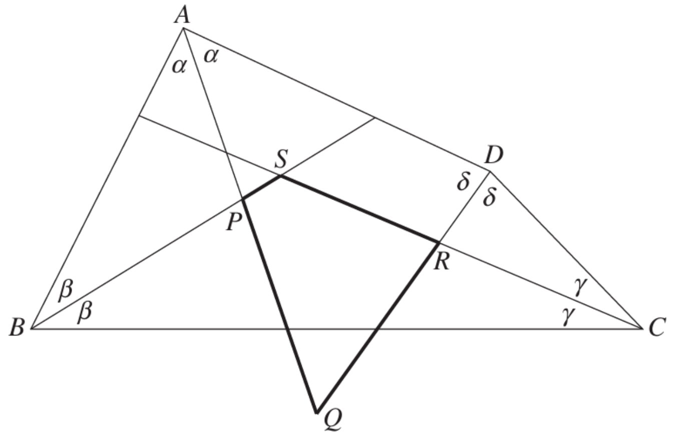
∴ ∠CDY = 70° ∠CAB = ∠CDB (Angles standing on same chord at the circumference are equal)

∴ ∠CAB = 70°

6 2018 HSC Extension 1 Band 5

The diagram shows quadrilateral ABCD and the bisectors of the angles at A, B, C and D.
The bisectors at A and B intersect at the point P. The bisectors at A and D meet at Q.
The bisectors at C and D meet at R. The bisectors at B and C meet at S.

Show that PQRS is a cyclic quadrilateral.



Let $\alpha = \frac{1}{2}\angle A, \beta = \frac{1}{2}\angle B$

$\gamma = \frac{1}{2}\angle C, \delta = \frac{1}{2}\angle D$

Then $2\alpha + 2\beta + 2\gamma + 2\delta = 360^\circ$ ($\angle$ sum quadrilateral)

$\therefore \alpha + \beta + \gamma + \delta = 180^\circ$

$\angle PSR = \angle BSC = 180^\circ - (\beta + \gamma)$ ($\angle$ sum of $\triangle$BSC)

$\angle PQR = \angle AQD = 180^\circ - (\alpha + \delta)$ ($\angle$ sum of $\triangle$AQD)

$\therefore \angle PSR + \angle PQR = 2 \times 180^\circ - (\alpha + \beta + \gamma + \delta)$

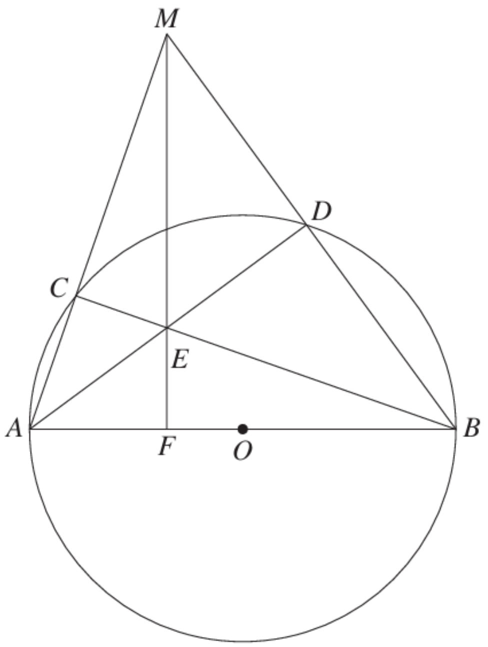
$= 2 \times 180° - 180°$

$= 180°$

∴ PQRS is cyclic (opposite angles are supplementary)

7 2016 HSC Extension 1 Band 5

The circle centred at O has a diameter AB. From the point M outside the circle the line segments MA and MB are drawn meeting the circle at C and D respectively.
The chords AD and BC meet at E. The line segment ME produced meets the diameter AB at F.



a Show that $CMDE$ is a cyclic quadrilateral.

$\angle ADB = 90°, \angle BCA = 90°$ ($\angle$ in a semicircle)

$\angle MDE = 90°$ ($\angle$ sum, straight line)

$\therefore$ $CMDE$ is a cyclic quadrilateral (Exterior $\angle$ACB equal to opposite interior $\angle$MDE)

b Hence, or otherwise, prove that MF is perpendicular to AB.

Construct CD

Let $\angle ABC = x$

$\therefore \angle ADC = x$ ($\angle$s in the same segment)

Similarly $\angle CME = x$

$\angle CAB = 90^\circ - x$ ($\angle$ sum $\triangle$ABC)

$\therefore \angle MFA = 180^\circ - \angle CME - \angle CAB$ ($\angle$ sum $\triangle$MFA)

$= 180° - x - (90° - x°)$

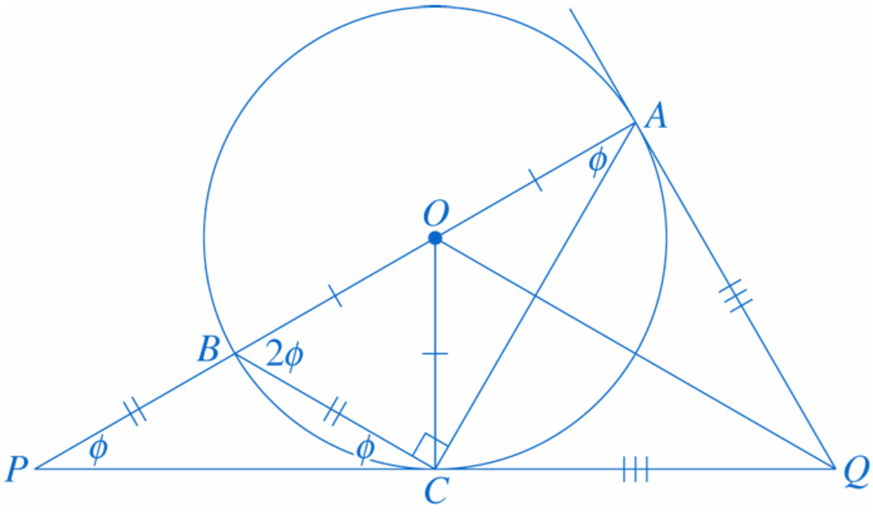
$= 90°$

$\therefore MF \perp AB$

8 2016 HSC Extension 2 Band 5

The circle centred at O has diameter AB. A point P on AB produced is chosen so that PC is a tangent to the circle at C and BP = BC. The tangents to the circle at A and C meet at Q.

Prove that $OP = OQ$.



Construct OC, AC, OQ

Let $\angle BPC$ have size ϕ
$\triangle BPC$ is isosceles so $\angle BCP$ is ϕ
$\angle OBC$ is an external angle
So $\angle OBC = 2\phi$

$\angle BAC$ is angle on chord BC
So $\angle BAC = \phi$ (Angle in alt. seg. and angle between chord and tangent)

But $\triangle BAC$ has a right angle at C (Angle in semicircle)

So $\phi + 2\phi = 90°$

$\phi = 30°$

$OA \perp AQ$ (Angle between tangent and radius)

So $\angle CAQ = 60°$

$QA = QC$ (Tangents from point equal)

So $\angle QCA = 60°$ ($\triangle QAC$ isosceles)

Hence $\angle AQC = 60°$ (Angle sum of $\triangle$)

$\triangle OAQ$ and $\triangle OCQ$ are cong. (sss, $OA = OC$ radii, $AQ = CQ$ above, OQ common)

So $\angle OQC = \dfrac{60°}{2} = 30° = \angle OPC$

Thus, $\triangle OPQ$ is isosceles
And, $OP = OQ$

MORE PROOFS

1 Prove that the length of a chord subtended by central angle θ is $d = 2r\sin\left(\frac{\theta}{2}\right)$

Consider chord AB subtended by angle $\theta = \angle AOB$

Draw the perpendicular from O to the midpoint of AB.

Using right angled trig in this triangle, $\sin\left(\frac{\theta}{2}\right) = \frac{\left(\frac{d}{2}\right)}{r}$