Student Name

Mathematics Stage 5 Paths

CIRCLE GEOMETRY

Book 1 Prove and apply angle and chord properties of circles

Version: 260207

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CONTENTS

OVERVIEW

SYLLABUS CONTENT

MA5-CIR-P-01 applies deductive reasoning to prove circle theorems and solve related problems

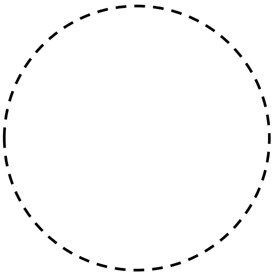
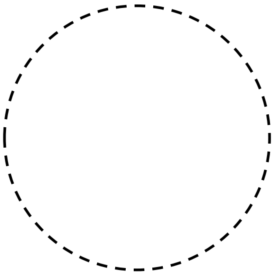
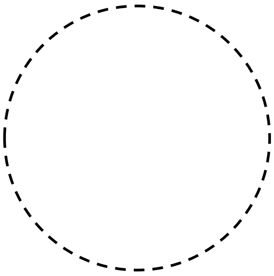
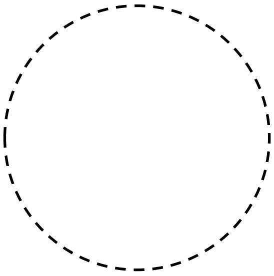
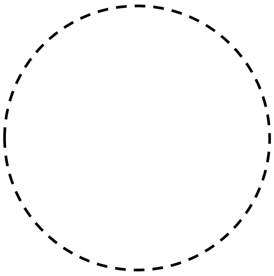
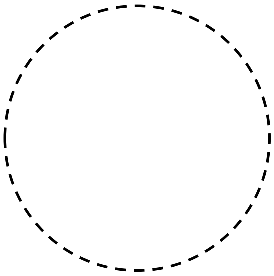
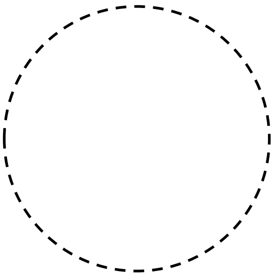
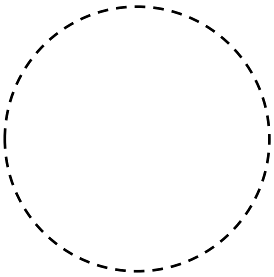
Prove and apply angle and chord properties of circles

- Apply terminology associated with angles in circles
- Identify the arc on which an angle at the centre or circumference stands
- Demonstrate that at any point on a circle there is a unique tangent to the circle, and that this tangent is perpendicular to the radius at the point of contact
- Prove chord properties of circles
- Prove angle properties of circles
- Apply chord and angle properties of circles to find unknown angles and lengths in diagrams

PARTS OF A CIRCLE

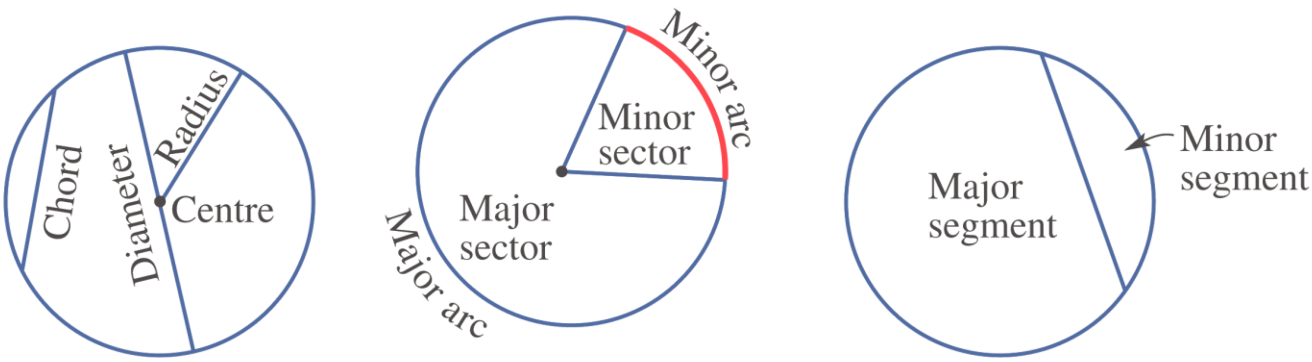
CIRCLE TERMINOLOGY

Investigation Parts of a circle.

Feature	Definition	Example (s)
Circumference	The circumference of a circle is the perimeter, or distance around the outside of a circle.	
Arc	An arc is a part of the circumference.	 
Centre	All points on the circumference is the same distance from the centre. The real definition of a circle is: a shape made by connecting all points an equal distance from the centre.	
Tangent	A tangent is a straight line that touches the circumference at a single point without entering the circle.	 
Chord	A chord is a straight line that goes from one point on the circumference to another point	 

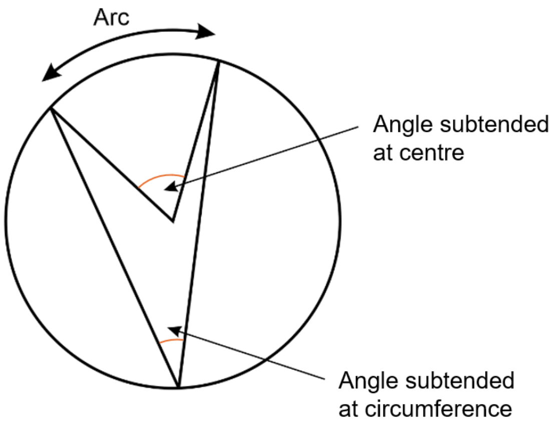
Diameter	The diameter is a special chord that goes through the centre of the circle.		
Radius	The radius is a line connecting the centre of a circle to the circumference. It is half the diameter.		
Segment	A segment is the area between a chord and the circumference.		
Sector	A sector is the area between two radii.		

SUMMARY



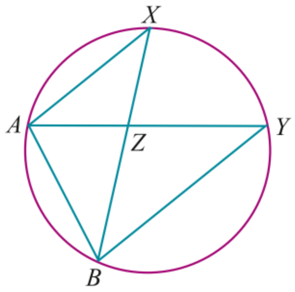
ARCS AND ANGLE AT THE CENTRE

- An arc/chord **subtends** (or forms) an angle at the centre and the circumference of a circle.
- When an angle is **subtended** by an arc, we say it's **standing** on that arc.



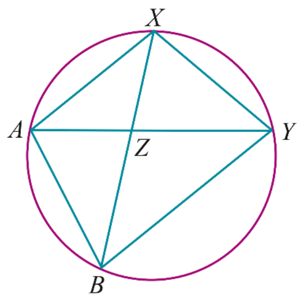
Example Identify angle subtended by a chord.

Name two angles subtended by chord AB at the circumference.



$\angle AXB$ and $\angle AYB$

a Refer to the diagram to answer these questions.



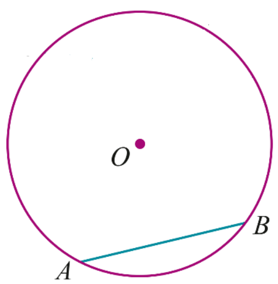
Name two angles subtended by chord XY at the circumference.

$\angle XAY$ and $\angle BBY$

Name two angles subtended by chord AY at the circumference.

$\angle AXY$ and $\angle ABY$

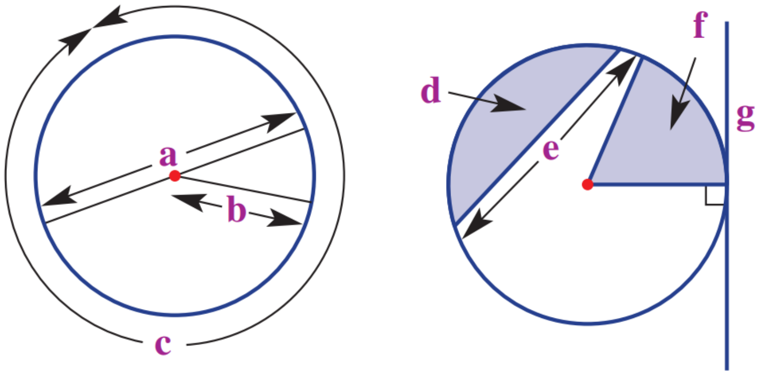
b Refer to the diagram. O is the centre of the circle. AB is a chord.



Draw the angle subtended at the centre by AB.

Draw an angle subtended at the circumference by AB.

1 Name the features of circles shown.



a

diameter

b

radius

c

circumference

d

segment

e

chord

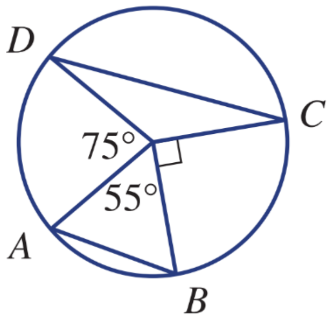
f

sector

g

tangent

2 Give the size of the angle in this circle subtended by the following:



a Chord AB

55°

b Minor arc BC

90°

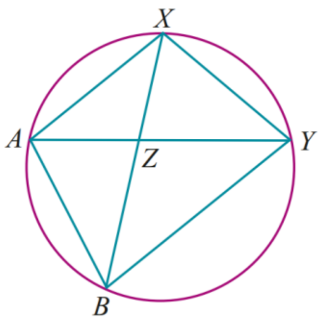
c Minor arc AD

75°

d Chord DC

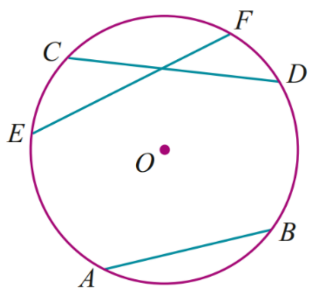
140°

3 Name two angles subtended by chord *BX* at the circumference.



$\angle XAB$ and $\angle XYB$

4 O is the centre of the circle.
Draw and name the angle subtended at the centre by each of these chords.



a CD

$\angle COD$

b EF

$\angle EOF$

5 Use each of the following words to fill in the sentences below. Some of the words may be used more than once.

diameter radius circumference chord tangent arc
sector major segment minor segment radii

a A line that goes from the centre of the circle to the edge is called …………………

radius

b The distance around the edge of a circle is called the …………………

circumference

c Any line that touches the circumference of a circle on the outside once is called a …………………

tangent

d A ………………… is a line that cuts the circle into 2 segments: the ………………… ……………… and the ………………… …………………

chord, major segment, minor segment

e An ………………… is the distance along the circumference between two points.

arc

f 2 radii cut out a ………………… of the circle

sector

g The ………………… is twice the …………………

Diameter, radius

h A ………………… is a ………………… if it passes through the centre of the circle.

Chord, diameter

i The area of the ………………… plus the area of the ………………… is equal to the area of the whole circle.

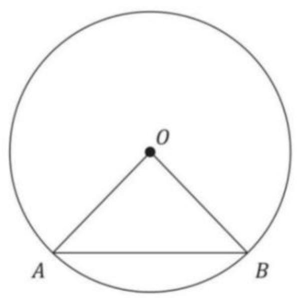
Minor segment, major segment

j A line that connects one point on the circumference to another is called a …………………

Chord

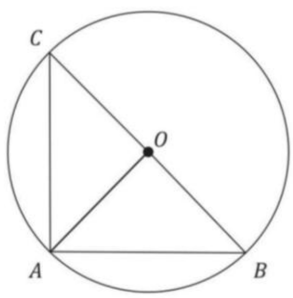
6 Identify the isosceles triangles in the diagrams below.

a



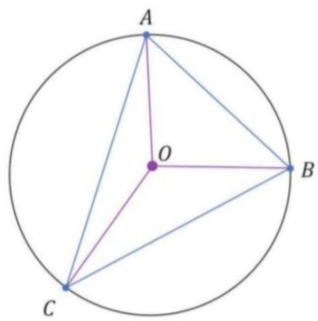
ΔAOB

b



ΔAOB and ΔCOA

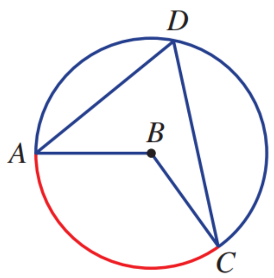
c



$\Delta AOB, \Delta BOC, \Delta AOC$

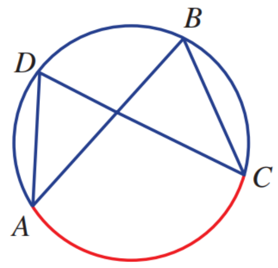
7 Name two angles subtended by the same arc AC in these circles.

a



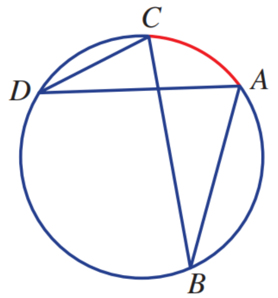
$\angle ABC, \angle ADC$

b



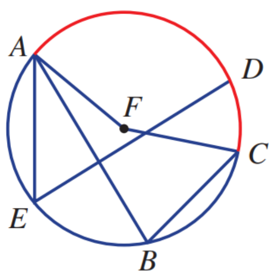
$\angle ADC, \angle ABC$

c



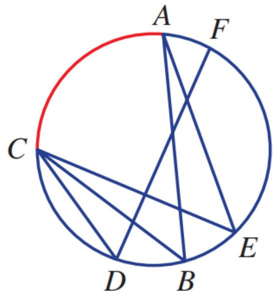
$\angle ABC, \angle ADC$

d



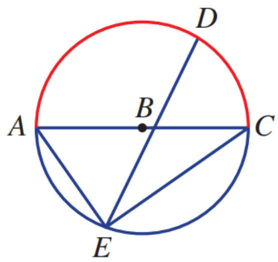
$\angle AFC, \angle ABC$

e



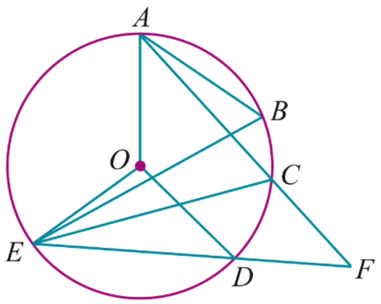
$\angle AEC, \angle ABC$

f



$\angle ABC, \angle AEC$

8 Name the chord that subtends each of these angles.



a $\angle EOD$ ED

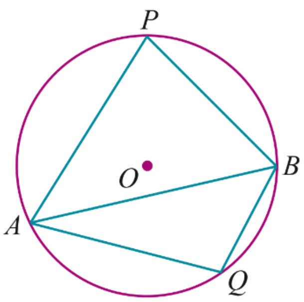
b $\angle CAB$ CB

c $\angle ABC$ AC

d $\angle EFA$ AE

e $\angle EDC$ EC

9 Name the angle subtended by chord AB at the circumference in:



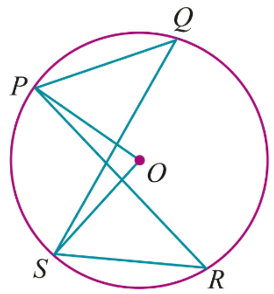
a The major segment

$\angle APB$

b The minor segment

$\angle AQB$

10



a Name the angle at the centre subtended by the arc PS.

$\angle POS$

b Name an angle at the circumference on the arc PS.

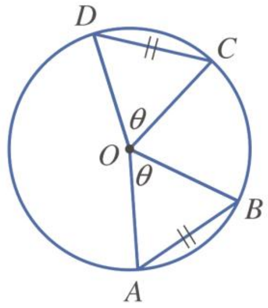
$\angle PQS$ or $\angle PRS$

c Name two angles standing on the arc QR.

$\angle QPR$ and $\angle QSR$

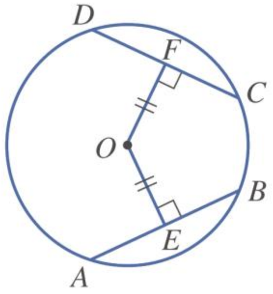
CHORD PROPERTIES

Equal chords subtend equal angles at the centre.



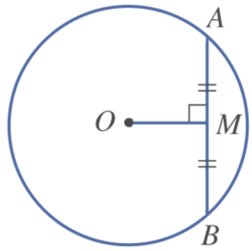
If AB = CD,
then ∠AOB = ∠COD

Equal chords are equidistant from the centre.



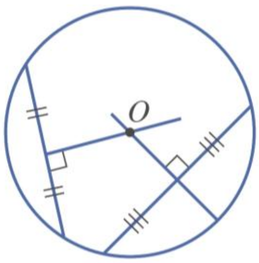
If AB = CD,
then OE = OF

The perpendicular from the centre of a circle bisects the chord.

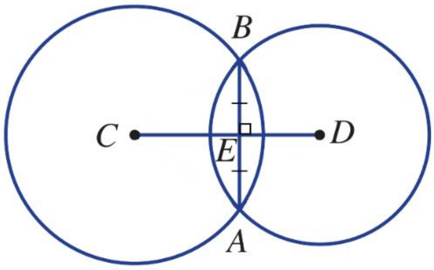


If OM ⊥ AB,
then AM = BM

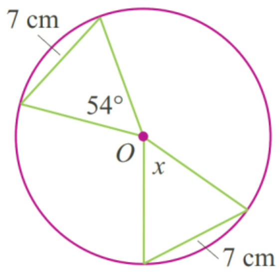
The perpendicular bisector of a chord passes through the centre.



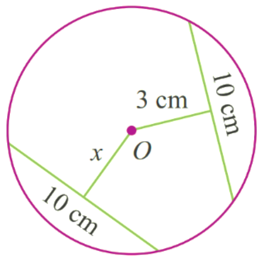
When 2 circles intersect, the line joining their centres bisects their common chord at right angles.



Example Apply chord properties to find unknown values.

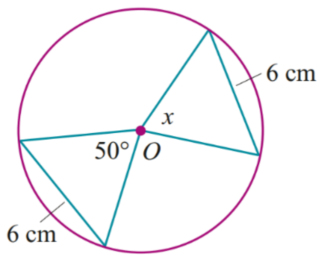


$x = 54°$
(Equal chords subtend equal angles at centre)



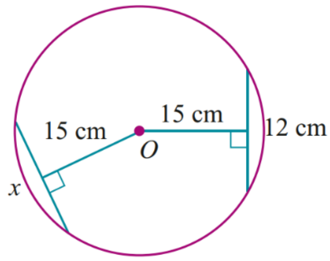
$x = 3$ cm
(Equal chords equidistant from centre)

a



50°

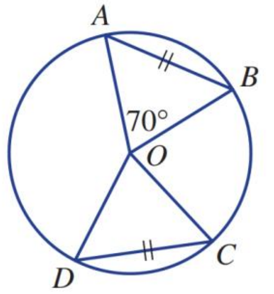
b



12 cm

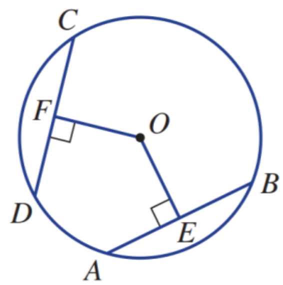
1 For each part, find the required information. You do not need to give reasoning.

a Given AB = CD and ∠AOB = 70°, find ∠DOC.



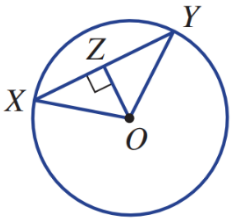
70°

b Given AB = CD and OF = 7.2 cm, find OE.



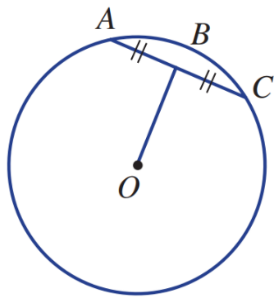
7.2 cm

c Given OZ ⊥ XY, XY = 8 cm and ∠XOY = 102°, find XZ and ∠XOZ.



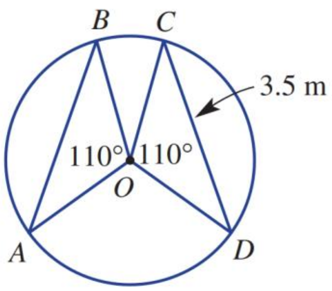
$XZ = 4$ cm
$\angle XOZ = 51°$

d Given AB = BC, find ∠OBC.



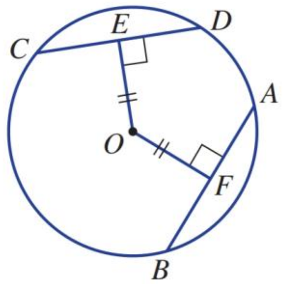
90°

e Given ∠AOB = ∠COD and CD = 3.5 m, find AB.



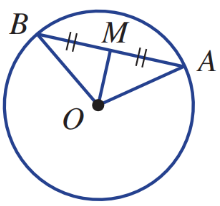
3.5 m

f Given OE = OF and AB = 9 m , find CD.



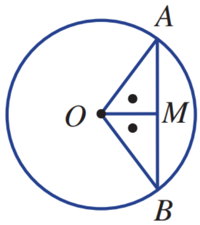
9 m

g Given M is the midpoint of AB, find ∠OMB.



90°

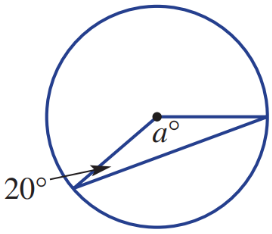
h Given ∠AOM = ∠BOM , find ∠OMB.



90°

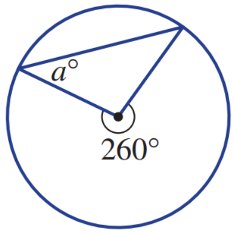
2 Find the size of each unknown value.

a



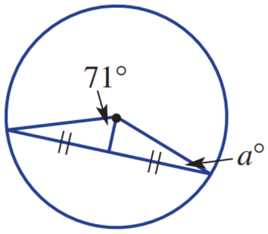
140°

b



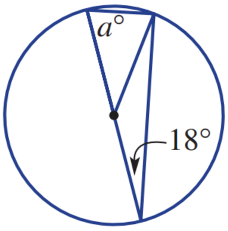
40°

c



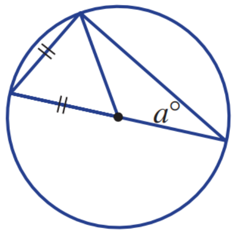
19°

d



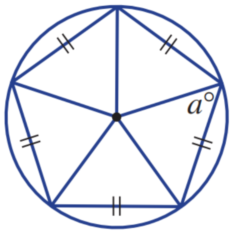
72°

e



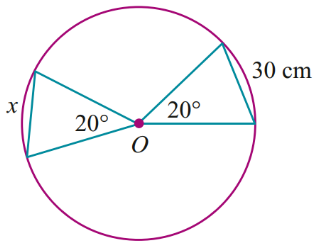
30°

f



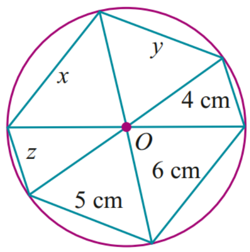
54°

g



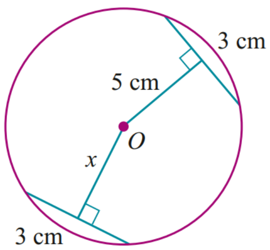
30 cm

h



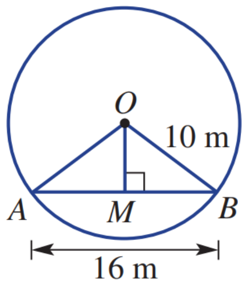
$x = 6, y = 5, z = 4$ cm

i



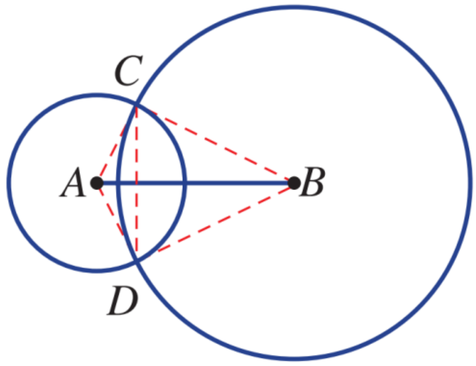
$x = 5$ cm

3 Find the length OM. Use Pythagoras' Theorem.



6 m

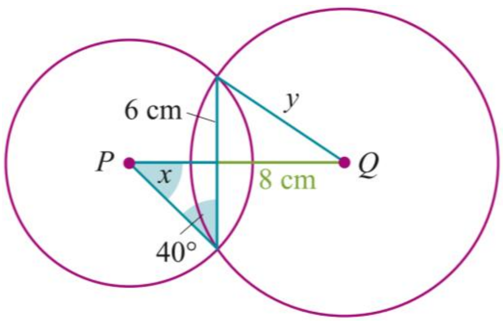
4 In this diagram, radius AD = 5 mm, radius BD = 12 mm and chord CD = 8 mm.
Find the exact length of AB, in surd form.



$3 + 8\sqrt{2}$ mm

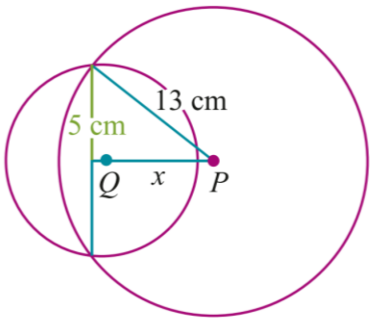
5 In the diagrams below, P and Q are the centres of the circles. Find the values of the variables.

a



$x = 50°, y = 10$ cm

b



$x = 12$ cm

6 Complete the following proofs.

a *Proof*: In $\triangle AOB$ and $\triangle COD$:

$AB = AC$	(given)
$AO = CO$	(radii)
$BO = DO$	(radii)
$\therefore \triangle AOB \equiv \triangle COD$	(SSS)
Hence $\angle AOB = \angle COD$	(Matching angles in congruent $\triangle$s are equal.)
Also $\angle ABO = \angle CDO$	(Matching angles in congruent $\triangle$s are equal.)

b *Proof*: In $\triangle OPB$ and $\triangle OQD$:

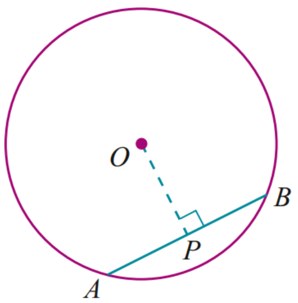
$\angle OPB = \angle OQD = 90°$	(given)
$\angle OBP = \angle ODQ$	(from part **a**)
$BO = DO$	(radii)
$\therefore \triangle OPB \equiv \triangle OQD$	(AAS)
$\therefore OP = OQ$	(Matching sides in congruent $\triangle$s are equal.)

From parts **a** and **b** we can conclude that:

Chords of equal length in a circle subtend equal angles at the centre and are equidistant from the centre.

c Use the diagram to prove $\triangle OAP \equiv \triangle OBP$, hence prove:

The perpendicular drawn from the centre of a circle to a chord bisects the chord.



$\angle OPA = \angle OPB = 90°$

$OA = OB$ (radii of a circle)

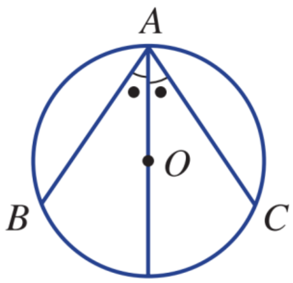
OP is common

$\triangle OAP \equiv \triangle OBP$ (RHS)

$\therefore AP = PB$ (corresponding sides of congruent triangles equal)

$\therefore$ The perpendicular drawn from the centre of a circle to a chord bisects the chord.

5 In this circle ∠BAO = ∠CAO. Prove AB = AC. Suggestion: Construct BC.

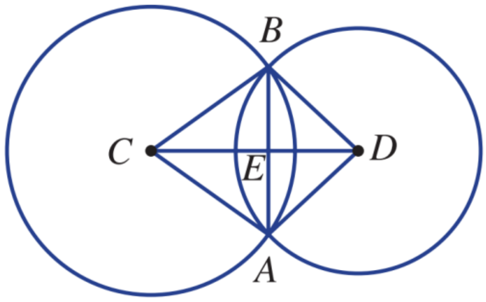


First, prove △OAB ≡ △OAC (AAS), which are isosceles.

So, AB = AC, corresponding sides in congruent triangles.

6 In this question we will prove the following theorem. When two circles intersect, the line joining their centres bisects that common chord at right angles.

a Prove △ACD ≡ △BCD.



AD = BD (equal radii),

AC = BC (equal radii),

CD is common.

Therefore, △ACD ≡ △BCD (SSS).

b Hence, prove △ACE ≡ △BCE.

AC = BC,

∠ACE = ∠BCE (corresponding angles),

CE is common.

Therefore, △ACE ≡ △BCE (SAS).

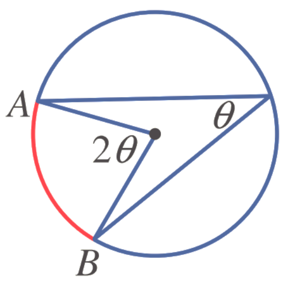
c Hence, prove CD ⊥ AB and BE = EA.

∠BEC = ∠CEA = 90° (equal angles on a straight line).

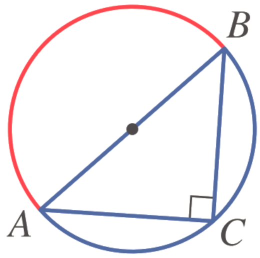
Therefore, CD ⊥ AB and BE = EA (matching sides in congruent triangles).

ANGLE PROPERTIES

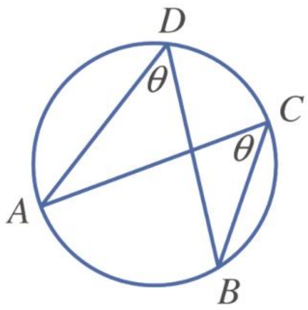
The angle at the centre is twice the angle at the circumference when standing on the same arc.



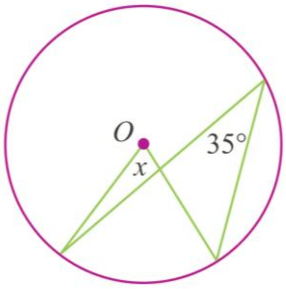
The angle in a semicircle is $90°$



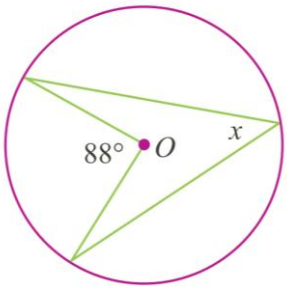
Angles at the circumference standing on the same arc are equal.



Example Apply angle properties to find unknown values.

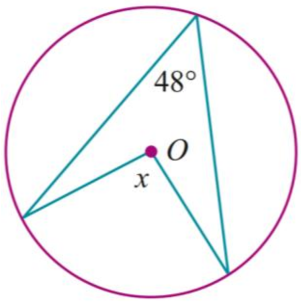


$x = 70°$ (Angle at centre twice angle at circumference when on the same arc.)



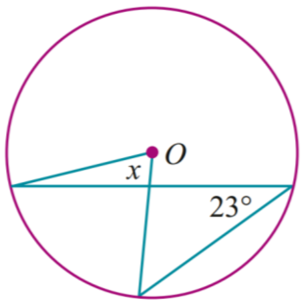
$x = 44°$ (angle at centre is twice angle at circumference when on the same arc.)

a



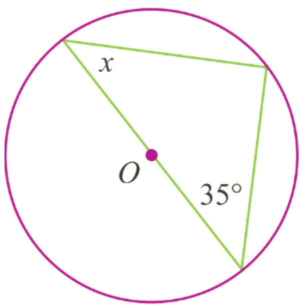
96

b



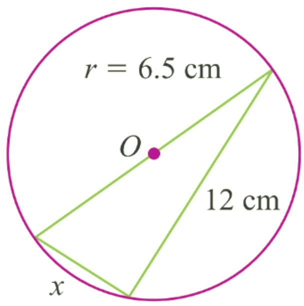
46

c



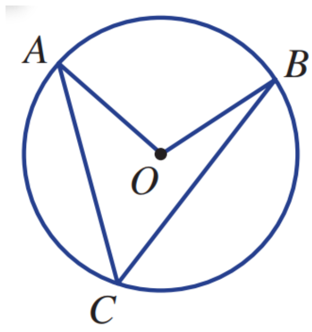
55

d



5 cm

1 For this circle, O is the centre.



a Name the angle at the centre of the circle subtended by AB.

∠AOB

b Name the angle at the circumference subtended by AB.

∠ACB

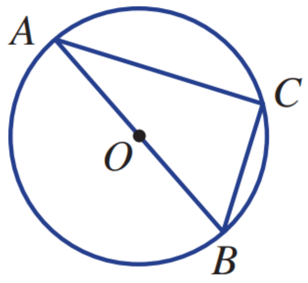
c If ∠ACB = 40°, find ∠AOB.

80°

d If ∠AOB = 122°, find ∠ACB.

61°

2 For this circle, AB is a diameter.



a What is the size of ∠AOB?

180°

b What is the size of ∠ACB?

90°

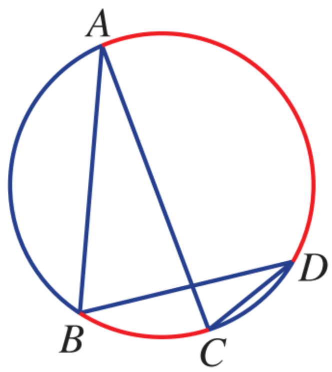
c If ∠CAB = 30°, find ∠ABC.

60°

d If ∠ABC = 83°, find ∠CAB.

7°

3 For this circle, answer the following.



a Name two angles subtended by arc AD.

∠ABD and ∠ACD

b State the size of ∠ACD if ∠ABD = 85°.

85°

c Name two angles subtended by arc BC.

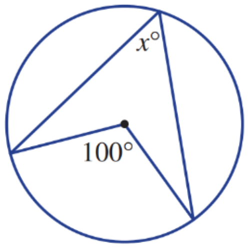
∠BAC and ∠BDC

d State the size of ∠BAC if ∠BDC = 17°.

17°

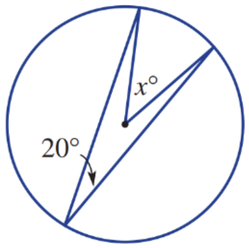
4 Find the value of x in these circles.

a



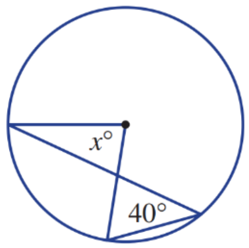
50°

b



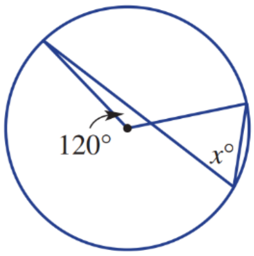
40°

c



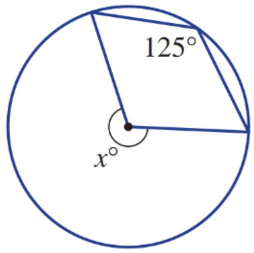
80°

d



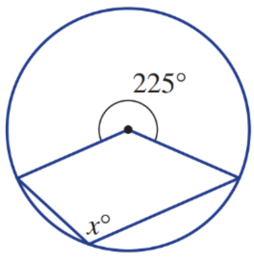
60°

e



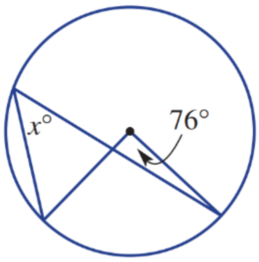
250°

f



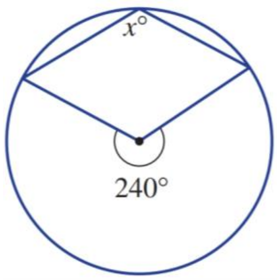
112.5°

g



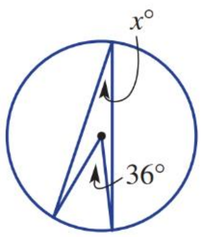
38°

h



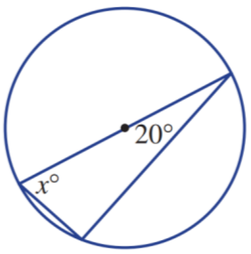
120°

i



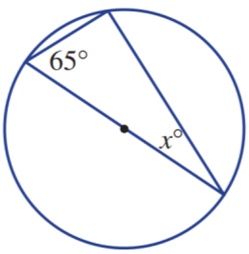
18°

j



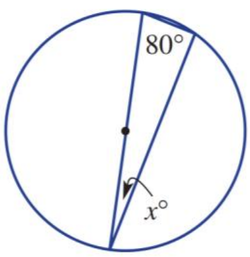
70°

k



25°

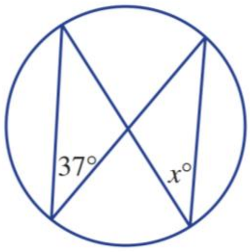
l



10°

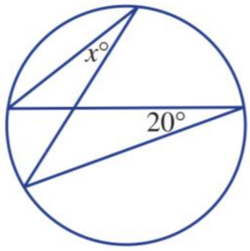
5 Find the value of x.

a



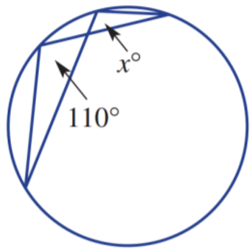
37°

b



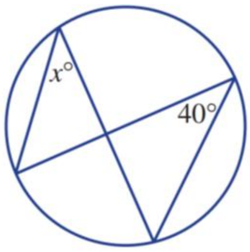
20°

c



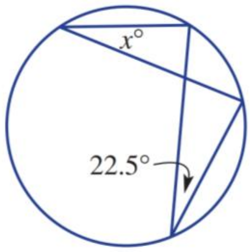
110°

d



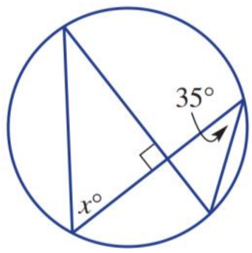
40°

e



22.5°

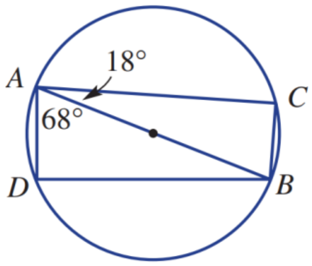
f



55°

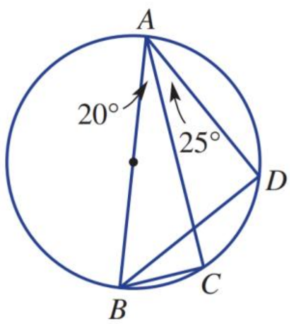
6 Find the size of both ∠ABC and ∠ABD.

a



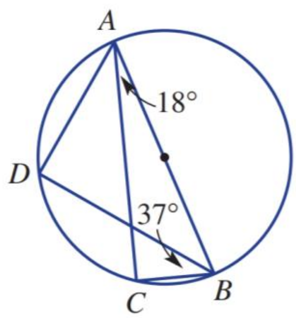
∠ABC = 72°, ∠ABD = 22°

b



∠ABC = 70°, ∠ABD = 45°

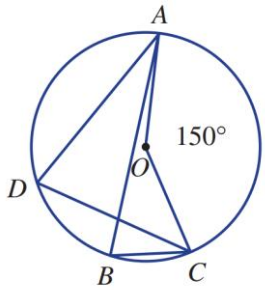
c



∠ABC = 72°, ∠ABD = 35°

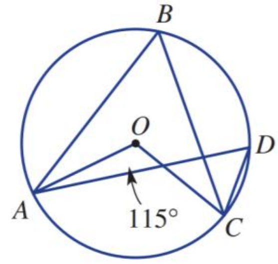
7

a Find ∠ADC and ∠ABC



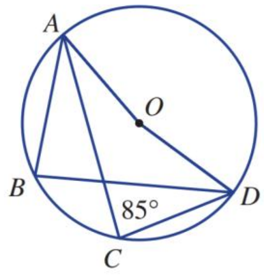
∠ADC = 75°, ∠ABC = 75°

b Find ∠ABC and ∠ADC



∠ABC = 57.5°, ∠ADC = 57.5°

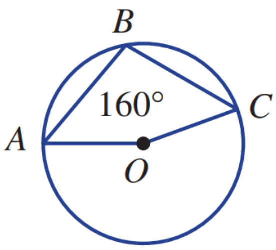
c Find ∠AOD and ∠ABD



∠AOD = 170°, ∠ABD = 85°

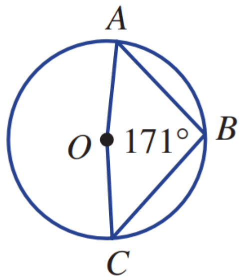
8 Find ∠ABC.

a



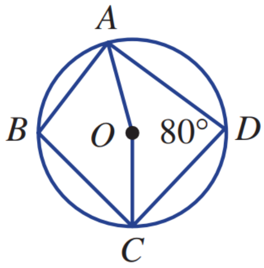
100°

b



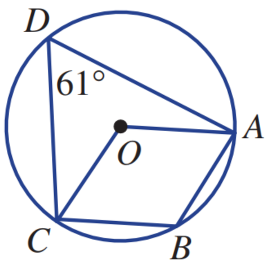
94.5°

c



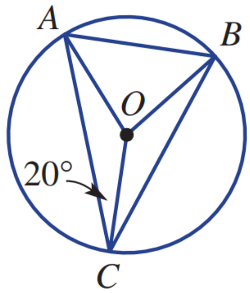
100°

d



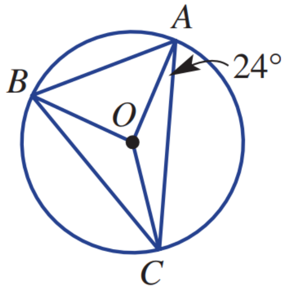
119°

e



70°

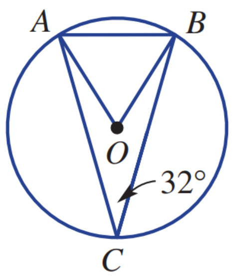
f



66°

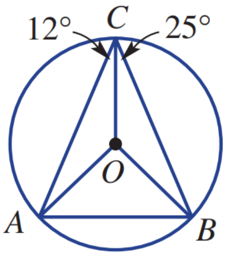
9 Find ∠OAB.

a



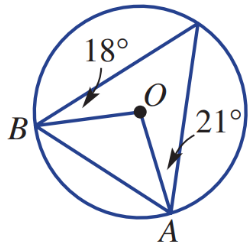
58°

b



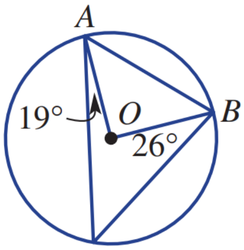
53°

c



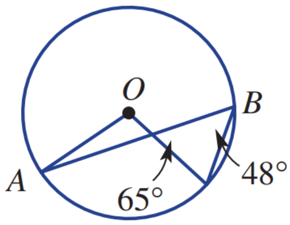
51°

d



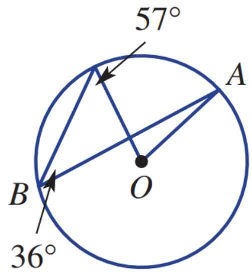
45°

e



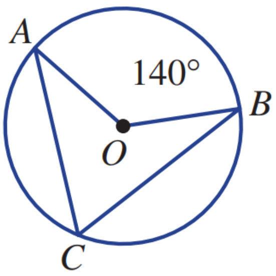
19°

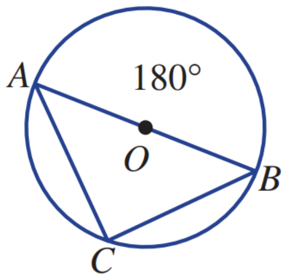
f



21°

10





a For the first circle shown, find ∠ACB

70°

b For the second circle shown, find ∠ACB

90°

c Explain why “The angle in a semicircle is 90°” is a special case of “The angle at the centre of a circle is twice the angle at the circumference when standing on the same arc.”

The second circle is the specific case of the first circle when the angle at the centre is 180°.

11 Complete the following proofs

a *Proof:* Let $\angle ACO = x$ and $\angle OCB = y$.

In $\triangle AOC$:

$\angle OAC = \angle OCA$ (Base angles of an isosceles $\triangle$ are equal; equal radii.)

$= x$

Now $\angle POA = \angle OAC + \angle OCA$

(The exterior angle of a $\triangle$ equals the sum of the interior opposite angles.)

$= 2x$

Similarly $\angle POB = 2y$

$\therefore \angle AOB = 2x + 2y = 2(x + y)$

$= 2 \times \angle ACB$

b

Proof: Let $\angle ACO = x$ and $\angle OCB = y$.

Show that $\angle POA = 2x$ and $\angle POB = 2y$.

Now $\angle AOB = \angle POB - \angle POA$

$= 2y - 2x = 2(y - x)$

But $\angle ACB = y - x$

$\therefore \angle AOB = 2 \times \angle ACB$

c *Proof:* $\angle APB = \frac{1}{2} \times \angle AOB$ and

(The angle at the centre of a circle is double the angle at the circumference, standing on the same arc.)

$\angle AQB = \frac{1}{2} \times \angle AOB$

(The angle at the centre of a circle is double the angle at the circumference, standing on the same arc.)

$\therefore \angle APB = \angle AQB$

d *Proof:* $\angle AOB = 2 \times \angle ACB$

(The angle at the centre of a circle is double the angle at the circumference, standing on the same arc.)

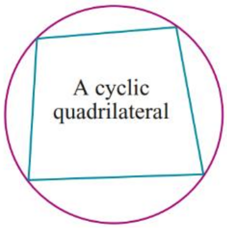
But $\angle AOB = 180°$ (Angles on a straight line equal 180°.)

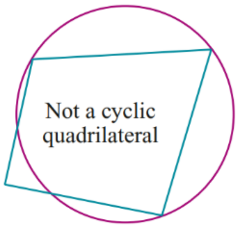
$2 \times \angle ACB = 180°$

$\therefore \angle ACB = 90°$

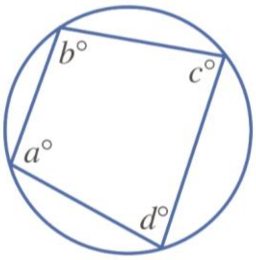
CYCLIC QUADRILATERALS

- A **cyclic quadrilateral** is a four-sided shape whose vertices all lie on a circle..



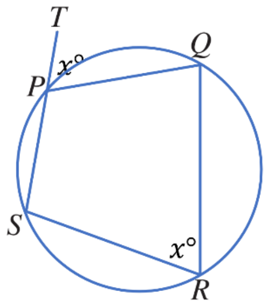


The opposite angles of cyclic quadrilaterals sum to 180° (supplementary).

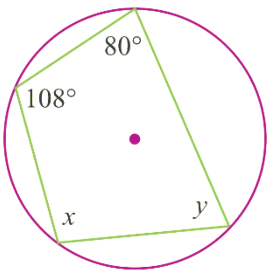


$$a + c = 180°$$
$$b + d = 180°$$

Exterior angles in cyclic quadrilaterals equal the opposite interior angle



Example Apply properties of cyclic quadrilaterals to find unknown values.

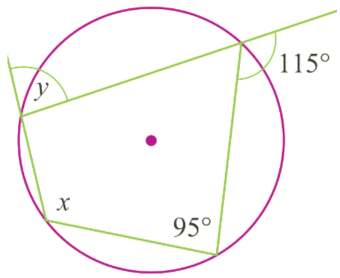


$x + 80° = 180°$

$x = 100°$

$108° + y = 180°$

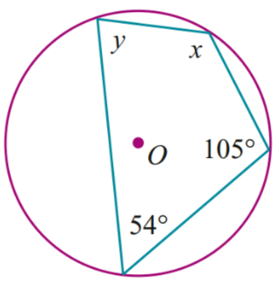
$y = 72°$



$x = 115°$

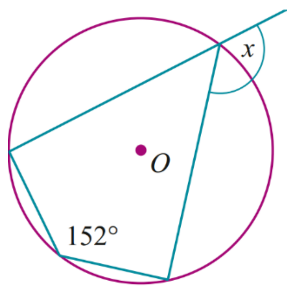
$y = 95°$

a



$x = 126, y = 75$

b

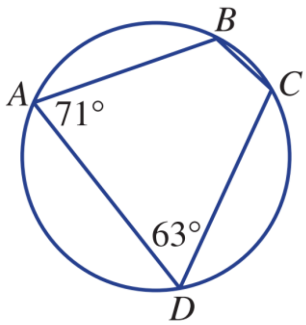


$x = 152$

1 Opposite angles in a cyclic quadrilateral are supplementary.

a What does it mean when we say two angles are supplementary?

Supplementary angles sum to 180°.



b Find ∠ABC.

117°

c Find ∠BCD.

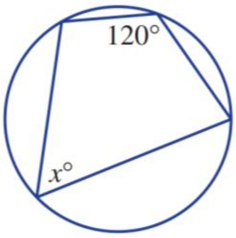
109°

d Check that ∠ABC + ∠BCD + ∠CDA + ∠DAB = 360°.

117° + 109° + 63° + 71° = 360

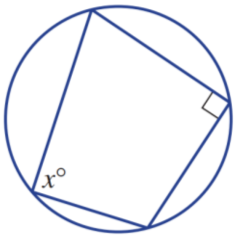
2 Find the value of the pronumerals in these circles.

a



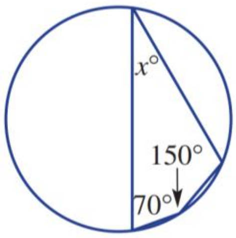
60°

b



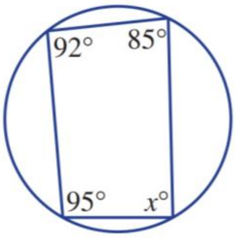
90°

c



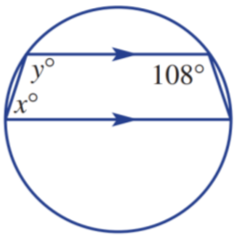
30°

d



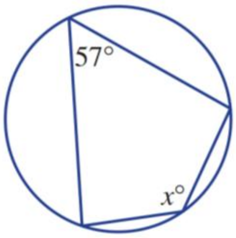
88°

e



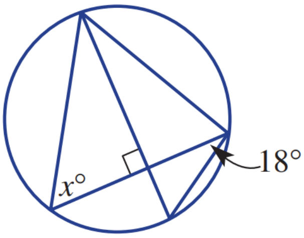
$x = 72°, y = 108°$

f



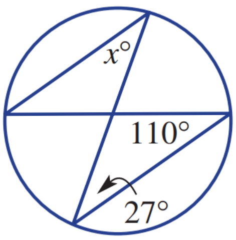
123°

g



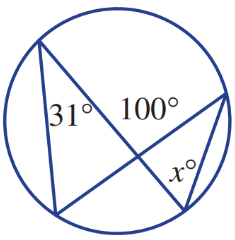
72

h



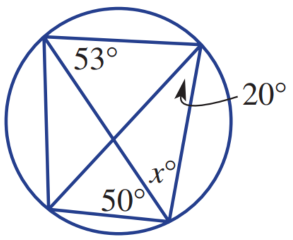
43

i



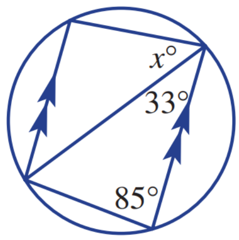
69

j



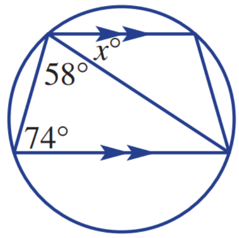
57

k



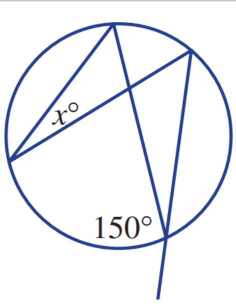
52

l



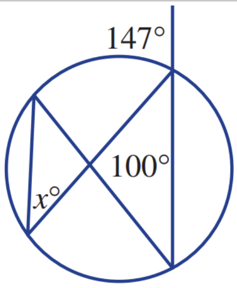
48

m



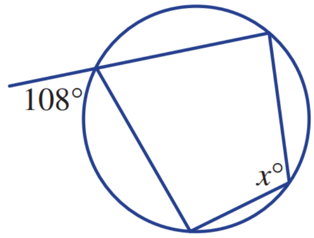
30

n



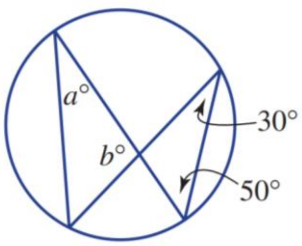
47

o



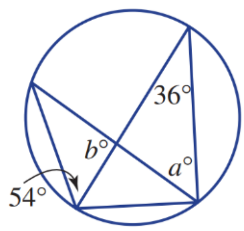
108

p



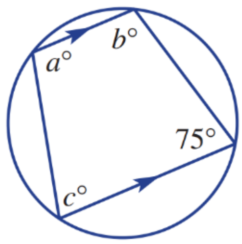
a = 30, b = 100

q



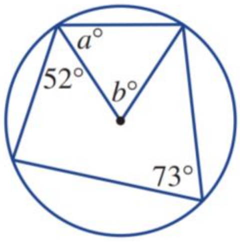
a = 54, b = 90

r



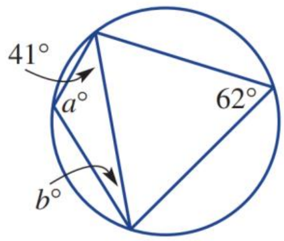
a = 105, b = 105, c = 75

s



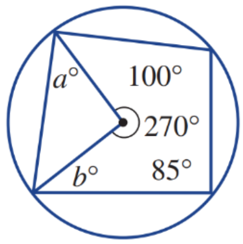
a = 55, b = 70

t



a = 118, b = 21

u



a = 45, b = 35

3 *Proof*: Let $\angle ABP = x$

$\angle ABC = 180° - x$ (CBP is a straight line.)

$\angle ADC = x$ (Opposite angles in a cyclic quadrilateral are supplementary.)

$\therefore \angle ABP = \angle ADC$.