Student Name

Mathematics Stage 5 Paths

POLYNOMIALS

Book 4 Graph polynomials

Version: 260207

Report mistakes & suggest improvements:
https://MrDingMaths.com

CONTENTS

OVERVIEW

SYLLABUS CONTENT

MA5-POL-P-01 defines, operates with and graphs polynomials and applies the factor and remainder theorems to solve problems

Graph polynomials

- Graph polynomials in factored form
- Graph quadratic, cubic and quartic polynomials by factorising and finding the zeroes
- Relate the term *zeroes* to polynomial functions and *roots* to polynomial equations
- Use graphing applications to determine the effect of single, double and triple roots of a polynomial equation $P(x) = 0$ on the shape of the graph for $y = P(x)$
- Graph polynomials using the sign of the leading term and the multiplicity of roots for the equation $P(x) = 0$
- Use graphing applications to compare the graphs of $y = -P(x)$, $y = P(-x)$, $y = P(x) + c$ and $y = kP(x)$ to the graph of $y = P(x)$

SKETCHING CUBICS IN FACTORED FORM

- A cubic function in factored form is $P(x) = k(x-a)(x-b)(x-c)$
  The x-intercepts are at $x = a,\ b,\ c$

1. Determine curve behaviour as $x \to \infty$.
   - If the coefficient of x^3 is positive, then as $x \to \infty$, $y \to \infty$.
   - If the coefficient of x^3 is negative, then as $x \to \infty$, $y \to -\infty$.
2. Determine the y-intercept by letting $x = 0$.
3. Determine the x-intercepts by letting $y = 0$.

Example Sketch cubic functions in factored form.

Sketch $y = (x+2)(x-1)(x-3)$

1. behaviour

Leading term is x^3,

$x \to \infty$, $y \to \infty$

2. y-int

Let $x = 0$

$y = (0+2)(0-1)(0-3)$

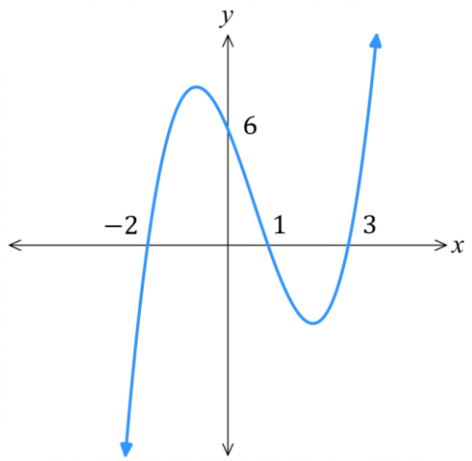
$= 6$

3. x-int

Let $y = 0$

$0 = (x+2)(x-1)(x-3)$

$x = 2, -1, -3$



Sketch $y = -x(x+3)(x-2)$

1. behaviour

Leading term is $-x^3$

so when $x \approx \infty$, $y \approx -\infty$

2. y-int

Let $x = 0$

$y = -0(0+3)(x-2)$

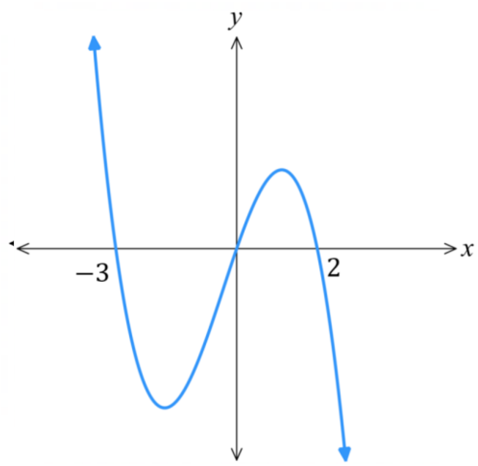
$= 0$

3. x-int

Let $y = 0$

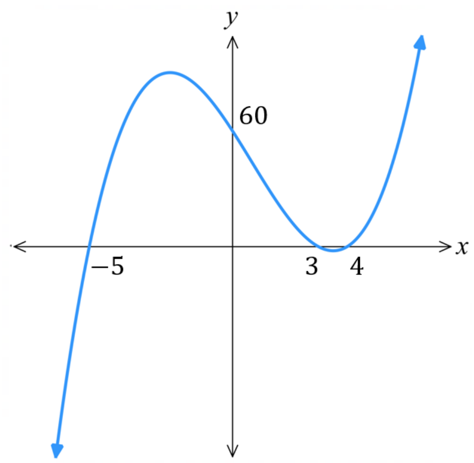
$0 = -x(x+3)(x-2)$

$x = 0, -3, 2$

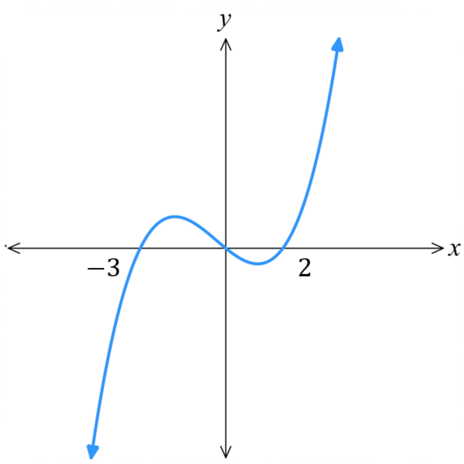


Sketch these cubic curves. Show the x and y-intercepts.

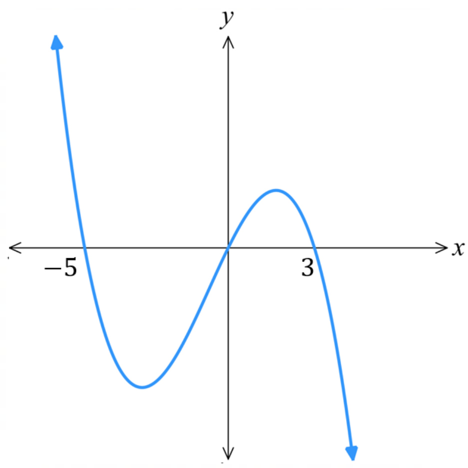
a $y = (x-4)(x+5)(x-3)$



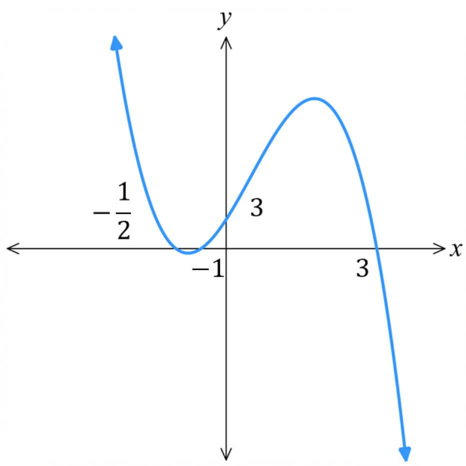
b $y = x(x-2)(x+3)$



c $y = -x(x+5)(x-3)$

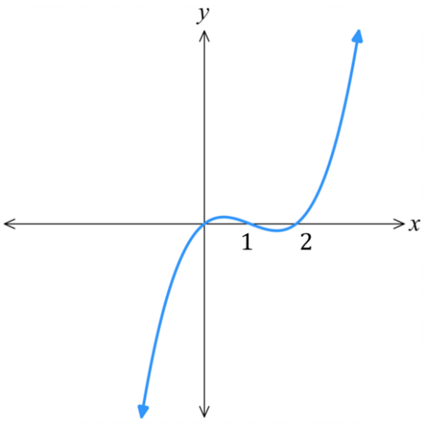


d $y = (3-x)(x+1)(2x+1)$

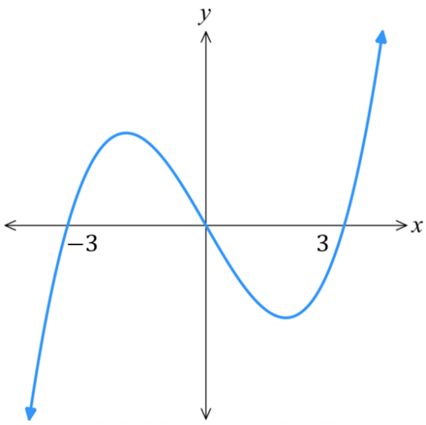


1 Sketch these cubic curves. Show the x and y-intercepts.

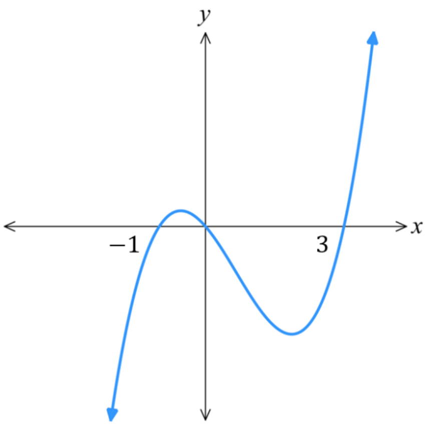
a $y = x(x - 1)(x - 2)$



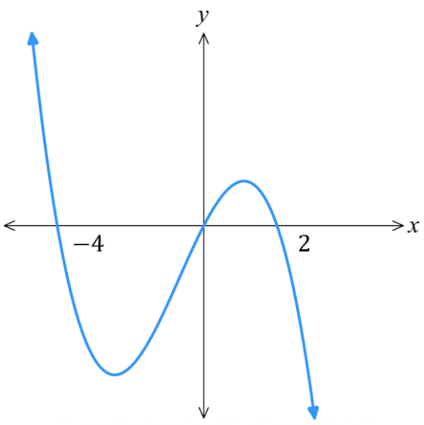
b $y = x(x - 3)(x + 3)$



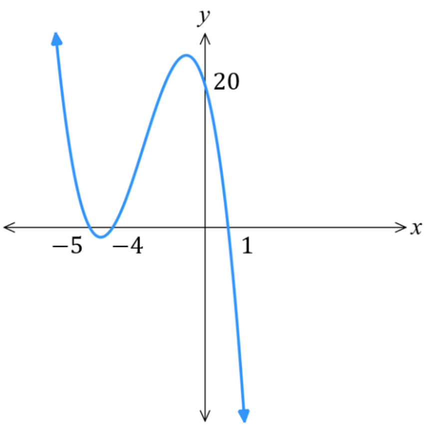
c $y = 2x(x - 3)(x + 1)$



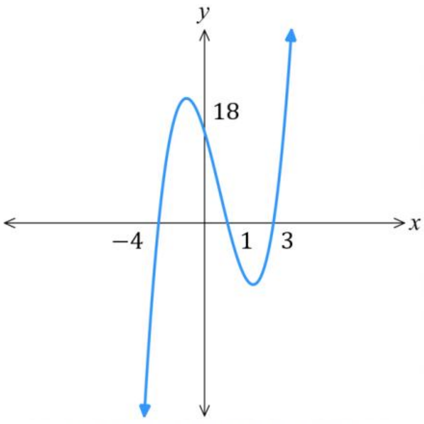
d $y = -x(x - 2)(x + 4)$



e $y = -(x + 4)(x - 1)(x + 5)$

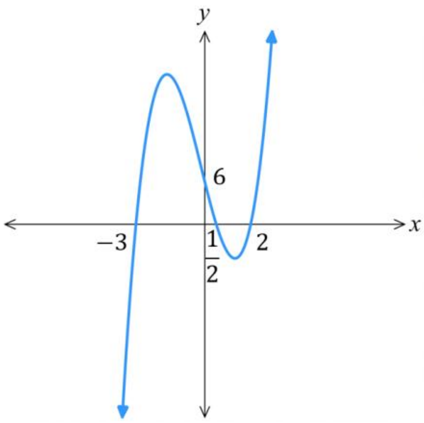


f $y = 3(x + 2)(x - 1)(x - 3)$

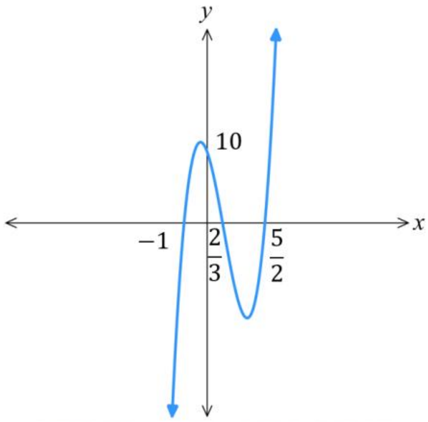


2 Sketch these cubic curves. Show the x and y-intercepts.

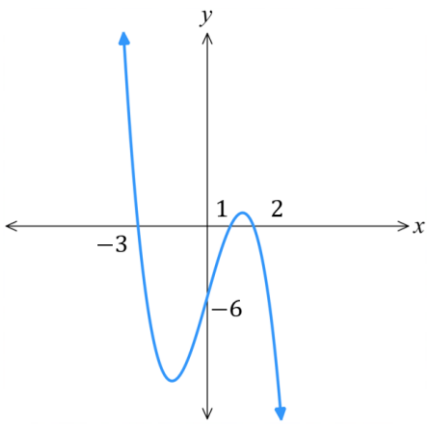
a $y = (2x-1)(x+3)(x-2)$



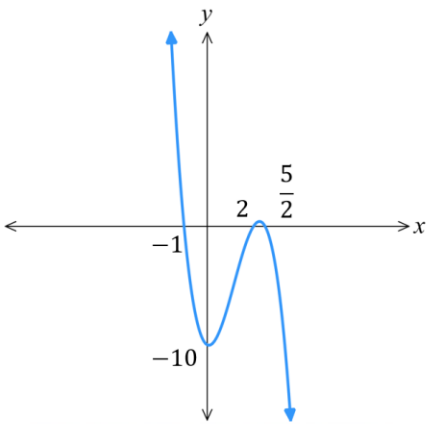
b $y = (3x-2)(x+1)(2x-5)$



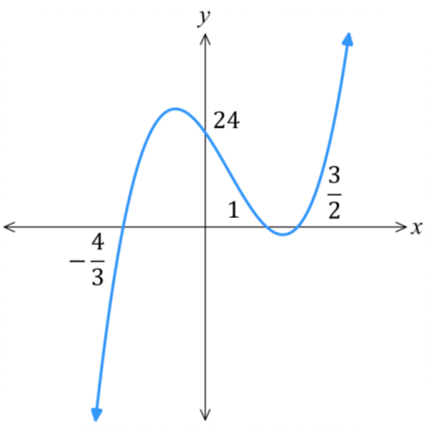
c $y = (1-x)(x+3)(x-2)$



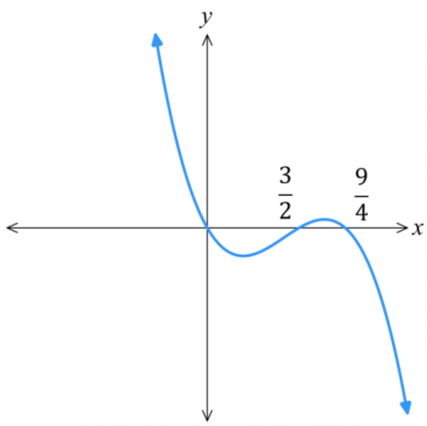
d $y = (2-x)(x+1)(2x-5)$



e $y = -2(3x+4)(2x-3)(1-x)$

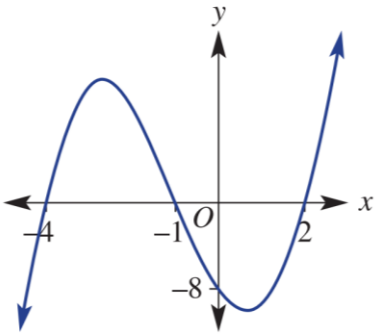


f $y = -x(9-4x)(3-2x)$



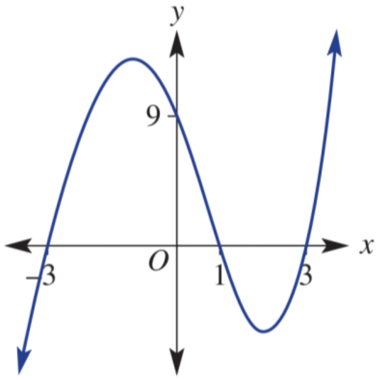
3 The following graphs are cubic curves. Find the equation using $y = k(x-\alpha)(x-\beta)(x-\gamma)$.

a



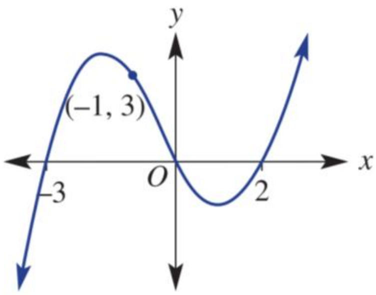
$y = (x-2)(x+1)(x+4)$

b



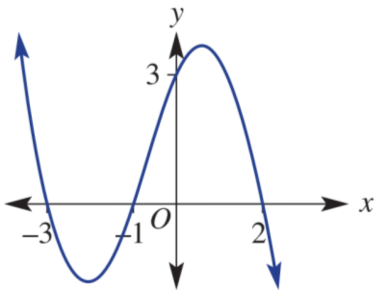
$y = (x+3)(x-1)(x-3)$

c



$y = \frac{1}{2}x(x-2)(x+3)$

d



$y = -\frac{1}{2}(x+3)(x+1)(x-2)$

4 A cubic function is in the form $y = k(x-\alpha)(x-\beta)(x-\gamma)$. Find its equation if:

a it has x-intercepts 2, 3 and −5 and passes through the point (−2, –120).

$y = -2(x-2)(x-3)(x+5)$

b it has x-intercepts −1, 4 and 6 and passes through the point (3, 96).

$y = 8(x+1)(x-4)(x-6)$

c it has x-intercepts 1 and 3, y-intercept −27 and k = −3.

$y = -3(x-1)(x-3)(x+3)$

GRAPHS OF POLYNOMIALS

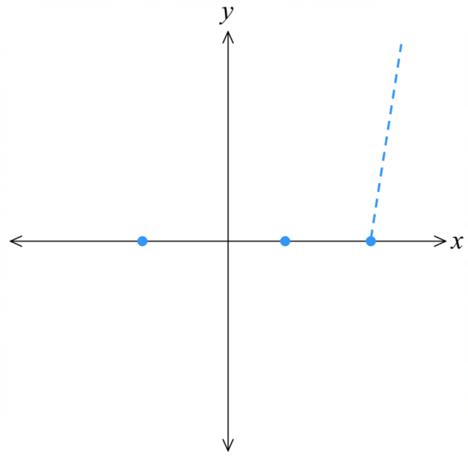
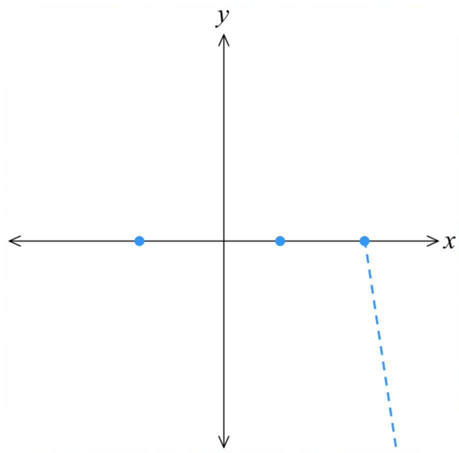
THE LEADING TERM AND CONSTANT TERM

Degree (highest power)

$$P(x) = \underbrace{\overbrace{a_n}^{\text{Leading coefficient}} x^{n}}_{\text{Leading term (term with the highest power)}} + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

Constant term

- The **leading term** determines the end behaviour of the graph.

Leading Term Positive	Leading Term Negative
As $x \to \infty$, $y \to \infty$ 	As $x \to \infty$, $y \to -\infty$ 

- The behaviour as $x \to -\infty$ will depend on the degree of the leading term.
- The **constant term** is the y-intercept.

- For a **factored polynomial:**
    - Find the leading term by multiplying the highest power terms from each factor.
    - Find the constant term by multiplying all constant terms from each factor.

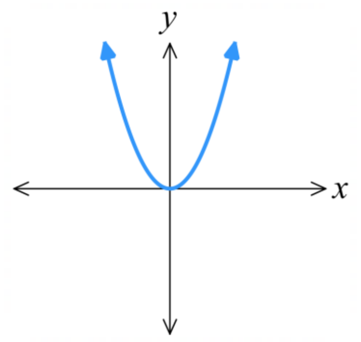
Example Identify leading term and constant term from a factored polynomial.

a $P(x) = (x-3)(x-4)(x+2)$ Leading term: Constant term:	**b** $P(x) = (2x-3)(x-4)(x+2)$ Leading term: Constant term:
c $P(x) = (2x-3)(4x-4)(x+2)$ Leading term: Constant term:	**d** $P(x) = -(2x-3)(4x-4)(x+2)$ Leading term: Constant term:
e $P(x) = -x(2x-3)(4x-4)(x+5)$ Leading term: Constant term:	**f** $P(x) = -2(2x-3)(4x-4)(x+5)$ Leading term: Constant term:
g $P(x) = (3-2x)(4x-4)(x+5)$ Leading term: Constant term:	**h** $P(x) = (3-2x)(4x-4)(5-x)$ Leading term: Constant term:
i $P(x) = x(x-1)^2$ Leading term: Constant term:	**j** $P(x) = 2x(x-1)^2$ Leading term: Constant term:
k $P(x) = -x(x-1)^2(x+3)^2$ Leading term: Constant term:	**l** $P(x) = (3x-1)^2(2-x)^3$ Leading term: Constant term:

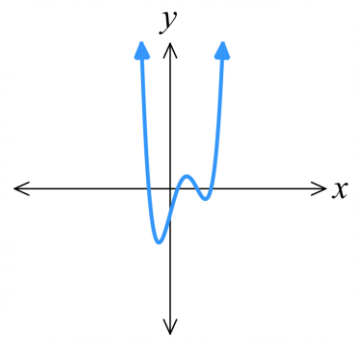
HOW THE LEADING TERM AFFECTS THE GRAPH

- If leading term has **even degree**, ends of graph point in the **same direction**.

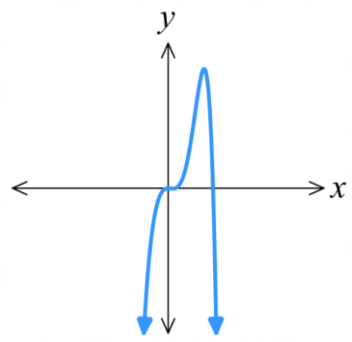
$y = x^2$



$y = 3x^4 - 6x^3 + 4x - 1$

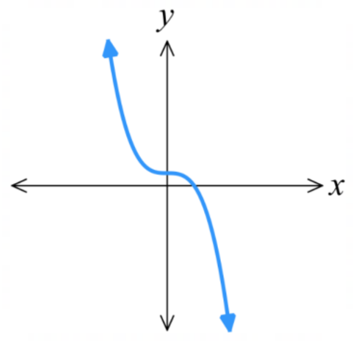


$y = -2x^6 + 11x^3 - 3x^2$

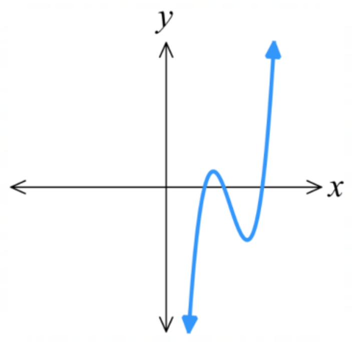


- If leading term has **odd** degree, ends of graph point in **different directions**.

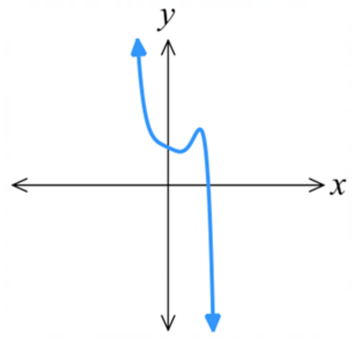
$y = 1 - x^3$



$y = x^3 - 10x^2 + 31x - 30$

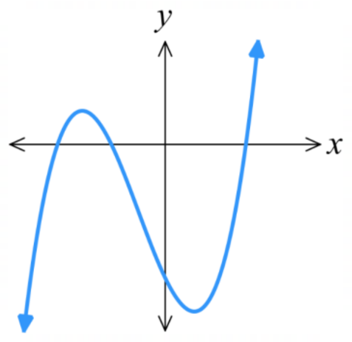


$y = -x^7 + 3x^4 - x + 3$

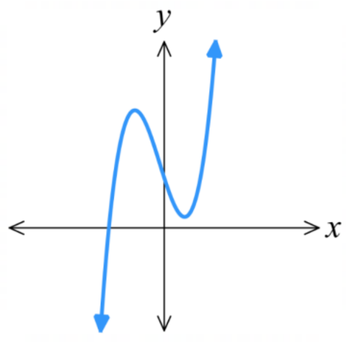


- If leading term is **positive**, as $x \to \infty, P(x) \to \infty$.

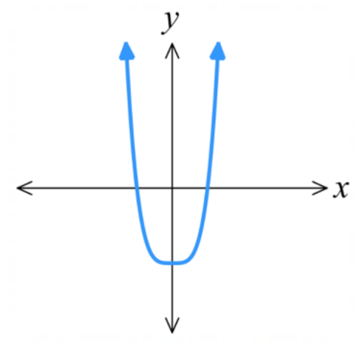
$y = x^3 + 3x^2 - 10x - 24$



$y = 2x^3 + x^2 - 5x + 3$

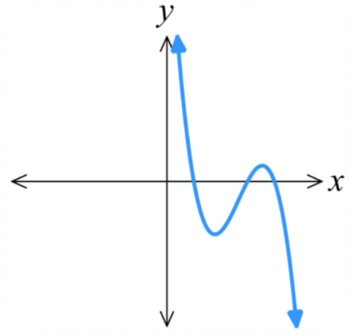


$y = x^4 - 3$

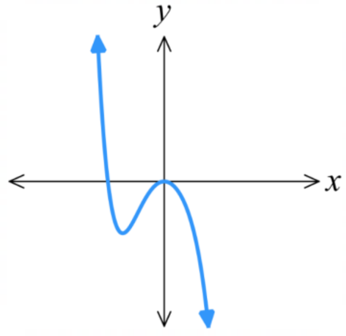


- If leading term is **negative**, as $x \to \infty, P(x) \to -\infty$.

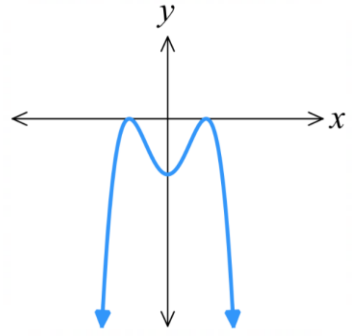
$y = -x^3 + 8x^2 - 19x + 12$



$y = -2x^5 - 4x^2$



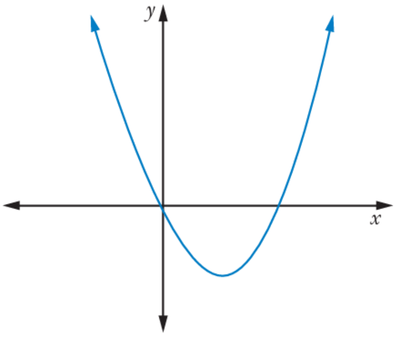
$y = 8x^2 - 16 - x^4$



Example Identify characteristics of leading term from graph.

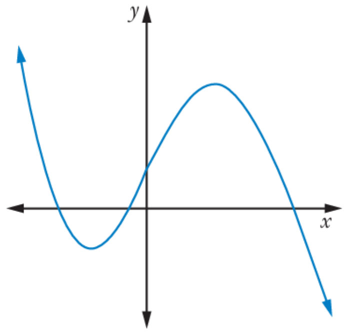
For each graph of a polynomial, determine whether the leading term is positive or negative, and whether the degree is odd or even.

a



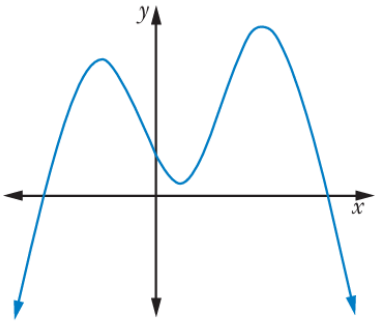
Leading term positive or negative? (circle)
Degree odd or even? (circle)

b



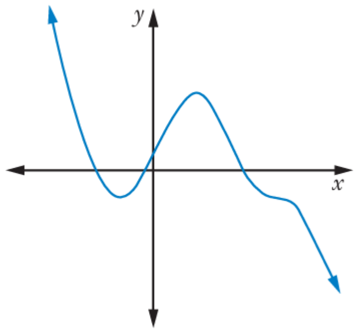
Leading term positive or negative?
Degree odd or even?

c



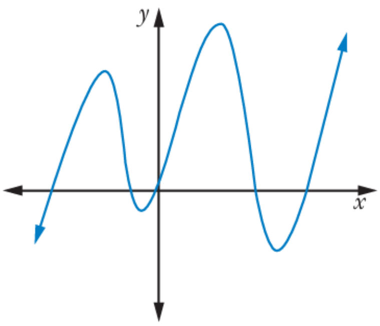
Leading term positive or negative?
Degree odd or even?

d



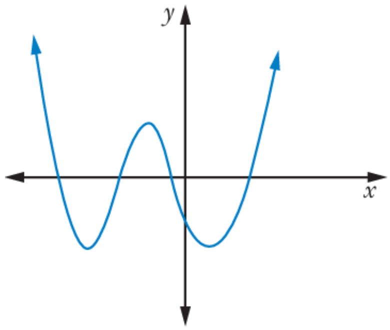
Leading term positive or negative?
Degree odd or even?

e



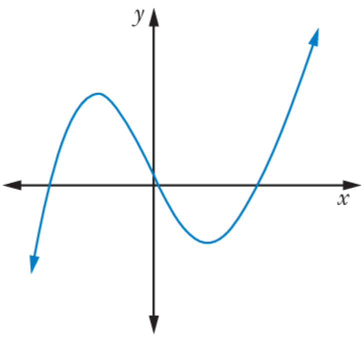
Leading term positive or negative?
Degree odd or even?

f



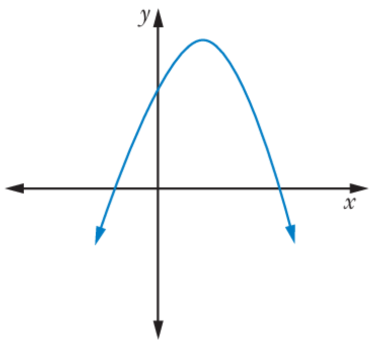
Leading term positive or negative?
Degree odd or even?

g



Leading term positive or negative?
Degree odd or even?

h



Leading term positive or negative?
Degree odd or even?

SKETCHING POLYNOMIALS

- A polynomial in factored form is $P(x) = k(x-a)(x-b)(x-c)(x-d)$ ...

  The x-intercepts are at $x = a,\ b,\ c,\ d \ldots$

1. Determine curve behaviour at $x \to \infty$ (identify leading coefficient).
   - If leading coefficient is positive, then $x \to \infty,\ y \to \infty$
   - If leading coefficient is negative, then $x \to \infty,\ y \to -\infty$
2. Determine the y-intercept by letting $x = 0$ (multiply constant terms in each factor).
3. Determine the x-intercepts by letting $y = 0$ (use null-factor law).

Example Sketch polynomials in factored form.

Sketch $y = (x-1)(x+3)(x-4)(x+2)$

1. behaviour

Leading term is x^4,

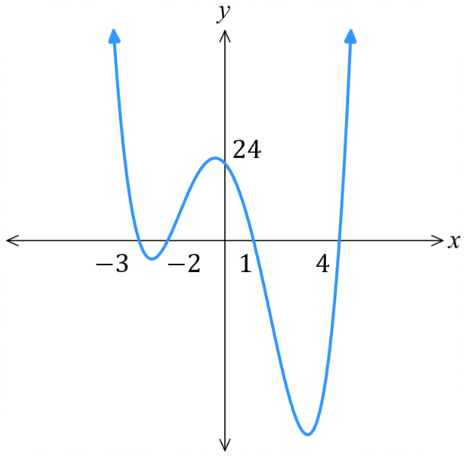
$x \to \infty,\ y \to \infty$

2. y-int

$y = (-1)(3)(-4)(2) = 24$

3. x-int

$x = 1, -3, 4, -2$



Sketch $y = -x(x-3)(2-x)(4-3x)(x+2)$

1. behaviour

Leading term is

$(-x)(x)(-x)(-3x)(x) = -3x^5$

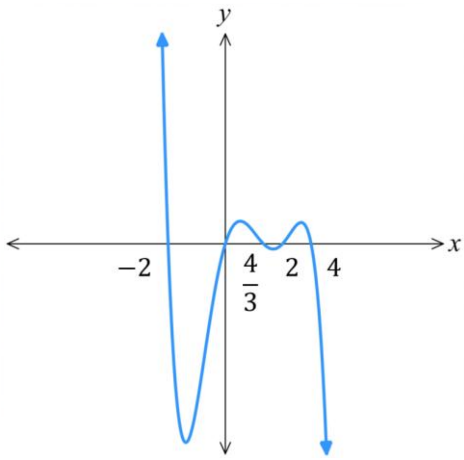
$x \to \infty,\ y \to -\infty$

2. y-int

$y = -0(-3)(2)(4)(2) = 0$

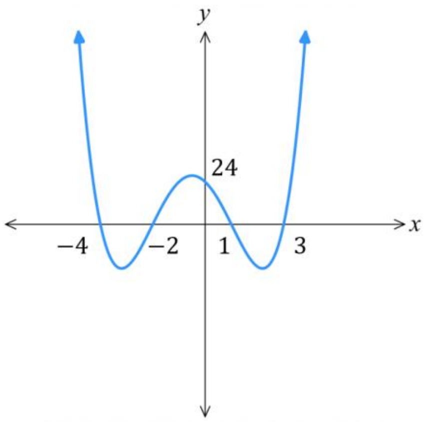
3. x-int

$x = 0, 3, 2, \frac{4}{3}, -2$

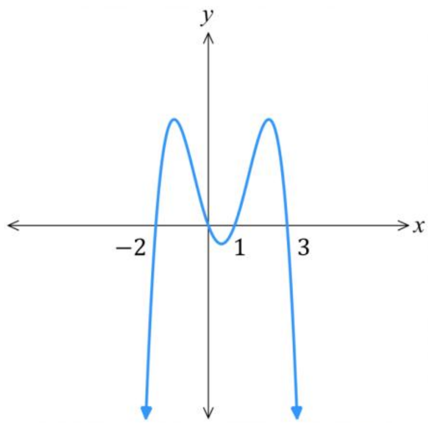


1 Sketch these polynomial curves. Show the x and y-intercepts.

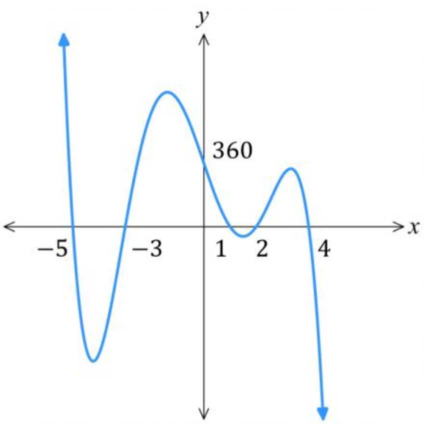
a $y = (x-3)(x+2)(x-1)(x+4)$



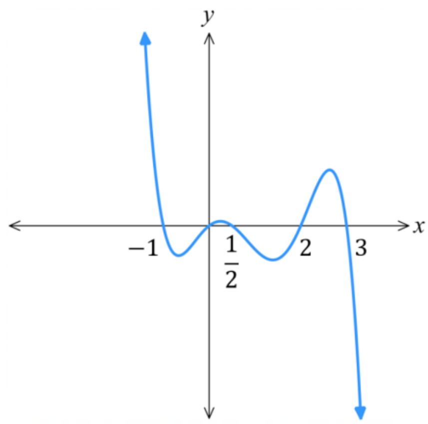
b $y = -2x(x-3)(x+2)(x-1)$



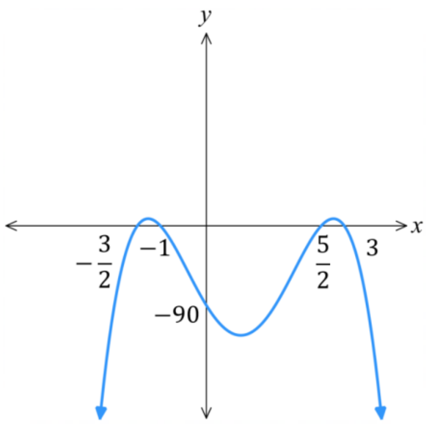
c $y = -3(x-2)(x+3)(x-4)(x+5)(x-1)$



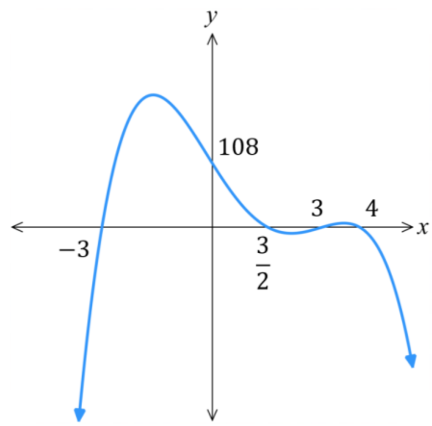
d $y = x(2x-1)(2-x)(x-3)(x+1)$



e $y = -2(2x+3)(x+1)(2x-5)(x-3)$

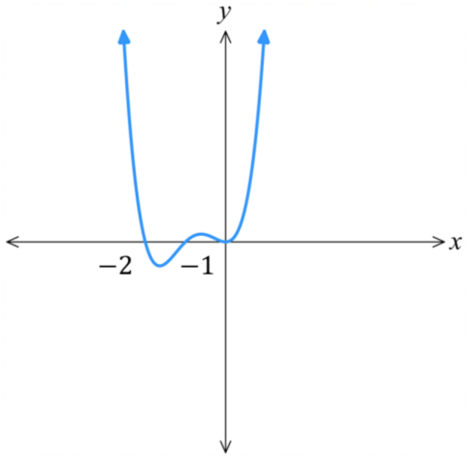


f $y = (3-2x)(4-x)(3-x)(x+3)$



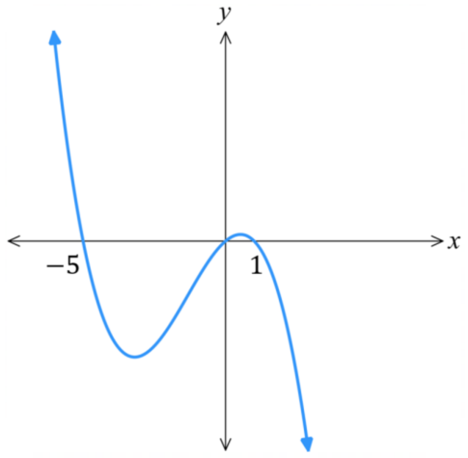
2 Sketch, by factorising first.

a $P(x) = x^4 + 3x^3 + 2x^2$



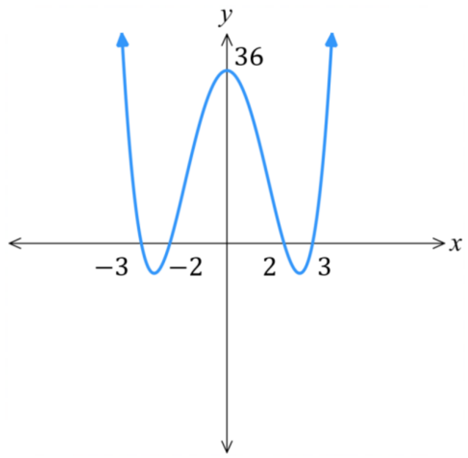
$P(x) = x^2(x+2)(x+1)$

b $P(x) = -x^3 - 4x^2 + 5x$



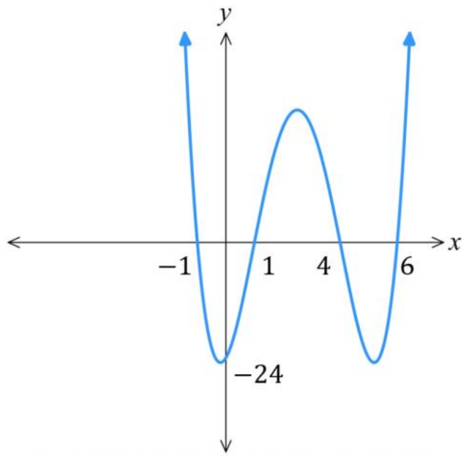
$P(x) = -x(x+5)(x-1)$

c $P(x) = x^4 - 13x^2 + 36$



$P(x) = (x-2)(x+2)(x-3)(x+3)$

d $P(x) = (x^2 - 5x)^2 - 2(x^2 - 5x) - 24$



$P(x) = (x+1)(x-6)(x-1)(x-4)$

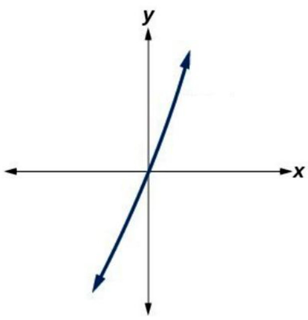
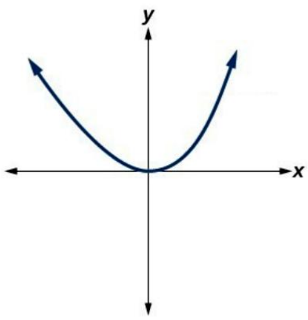
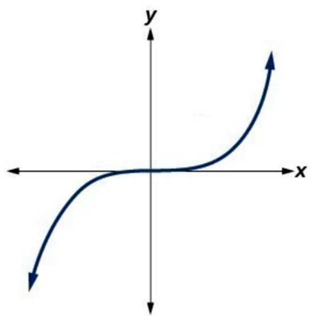
MULTIPLICITY OF ROOTS

- A **repeated factor** in a polynomial changes the curve's behaviour at the x-intercepts.
- For example, if a polynomial has a double root at $x = 5$, also called **root of multiplicity 2**, then the curve "bounces" off the x-axis like a parabola.

Investigation Multiplicity of roots and the curve at the x-intercept.

https://www.desmos.com/calculator

Sketch: $y = (x-a)$ $y = (x-a)^2$ $y = (x-a)^3$	Explain what happens to the curve at the x-intercept when there are repeated roots. …………………………………………………… …………………………………………………… ……………………………………………………
Sketch: $y = (x-1)(x-2)(x+3)$ $y = (x-1)(x-2)^2(x+3)$ $y = (x-1)(x-2)^2(x+3)^2$ $y = (x-1)(x-2)^2(x+3)^2$ $y = (x-1)(x-2)^2(x+3)^3$ $y = (x-1)^3(x-2)^2(x+3)^3$	Do the graphs match your observations above? …………………………………………………… …………………………………………………… ……………………………………………………

Multiplicity of Roots		
Single Root The curve "cuts" the axis like a straight line. 	**Double Root** The curve "bounces" off the axis like a parabola. 	**Triple Root** The curve inflects along the axis like a cubic curve. 

SKETCHING POLYNOMIALS WITH ROOTS OF MULTIPLICITY

1. Determine curve behaviour at $x \to \infty$ (identify leading coefficient).
   - If leading coefficient is positive, then $x \to \infty$, $y \to \infty$
   - If leading coefficient is negative, then $x \to \infty$, $y \to -\infty$
2. Determine the y-intercept by letting $x = 0$ (multiply constant terms in each factor).
3. Determine the x-intercepts by letting $y = 0$ (use null-factor law).
4. Identify the behaviour of the curve at the x-intercepts.

Example Sketch polynomials with repeated roots.

Sketch $y = (x-3)(x+1)^2$

1. behaviour

   Leading term is x^3

   $x \to \infty, y \to \infty$

2. y-int

   $y = (-3)(1)^2 = -3$

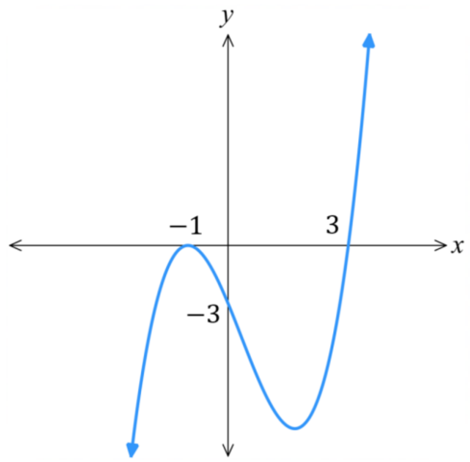
3. x-int

   $x = 3, -1$

4. behaviour at roots

   Cut at $x = 3$

   Bounce at $x = -1$



Sketch $y = -(x+1)^3(x-1)(x-3)$

1. behaviour

   The leading term will be $-x^5$

   so when $x \approx \infty$, $y \approx -\infty$

2. y-int

   $y = -(1)^3(-1)(-3) = -3$

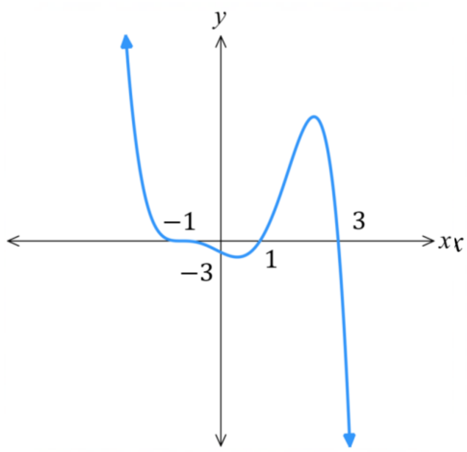
3. x-int

   $x = -1, 1, 3$

4. behaviour at roots

   Inflect at $x = -1$

   Cut at $x = 1,3$



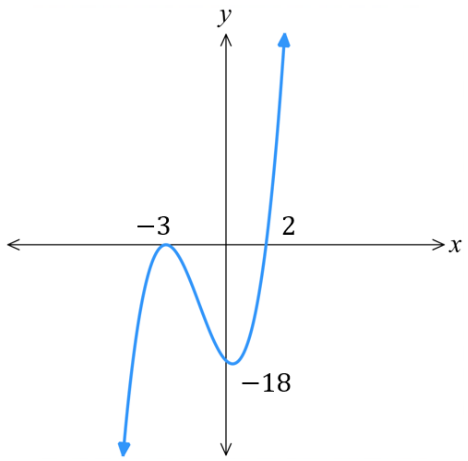
How do we know the behaviour of the curve at the x-intercepts (cut, bounce, or inflect)?

When a polynomial is written in factored form, if a factor has a power of ……, it cuts the axis.

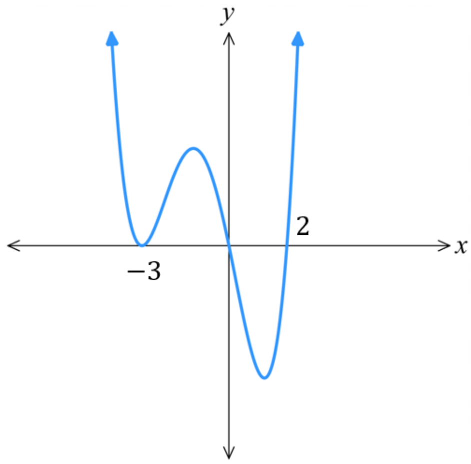
If a factor has a power of 2, it ……………, and if the factor has a power of ……, it inflects.

Sketch these polynomial functions.

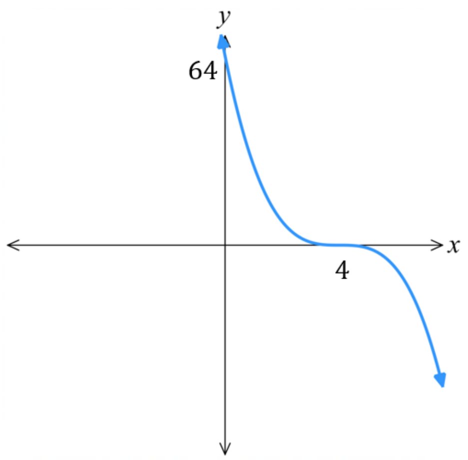
a $y = (x-2)(x+3)^2$



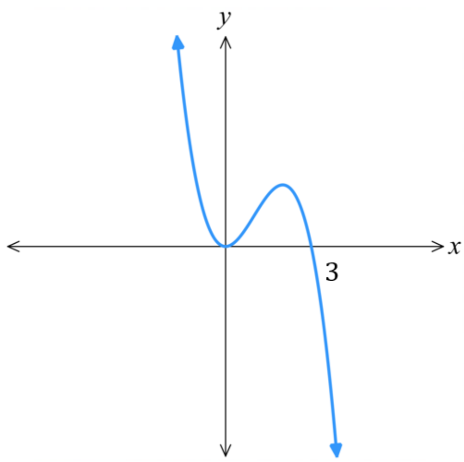
b $y = x(x-2)(x+3)^2$



c $y = -(x-4)^3$



d $y = 2x^2(3-x)$



3 For each of the following polynomials, state the roots and categorise the root as either: a single root, double root, or triple root.

a $P(x) = (x-1)(x+2)(x-3)^2$ $x = 1,$ single root $x = -2,$ ……… root $x = 3,$ ……… root	**b** $P(x) = (x+1)(x-3)(x+5)$
c $P(x) = (x-5)^2$	**d** $P(x) = (x+4)^3$
e $P(x) = (x-2)^2(x+4)$	**f** $P(x) = (x+7)^3(x+3)$
g $P(x) = (x+7)^3(x+3)^2(x-1)$	**h** $P(x) = (x^2-1)^2$ (Hint: the factor must be linear; factorise first)

4 Write down, in factorised form, the monic polynomial that has

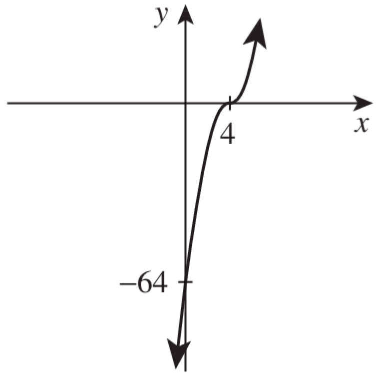
a A single root at $x = 1, -2,$ and 3 ……………………………………………

b A single root at $x = \pm 1$ and a double root at $x = 4$ ……………………………………………

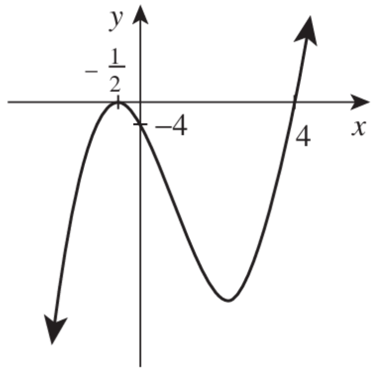
c A double root at $x = 2$ and a triple root at $x = -2$ ……………………………………………

5 Sketch these functions.

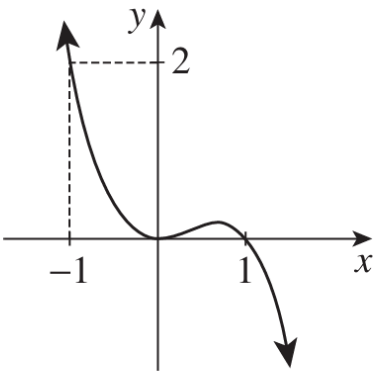
a $y = (x-4)^3$



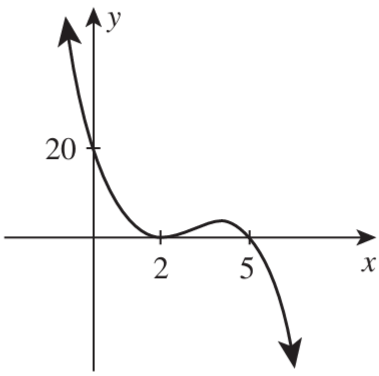
b $y = (2x+1)^2(x-4)$



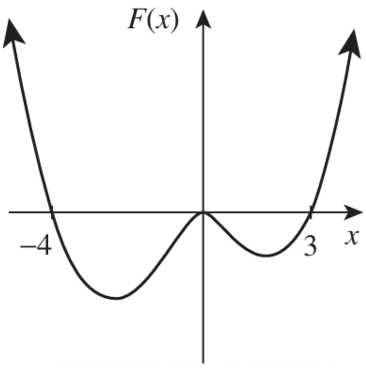
c $y = x^2(1-x)$



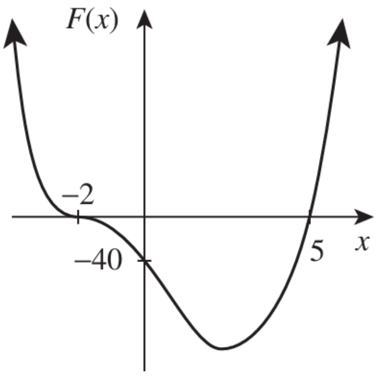
d $y = (2-x)^2(5-x)$



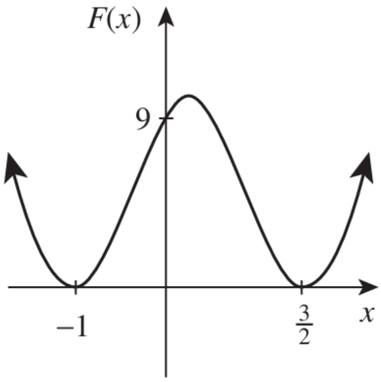
e $y = x^2(x+4)(x-3)$



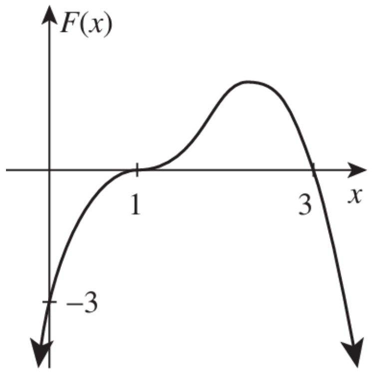
f $y = (x+2)^3(x-5)$



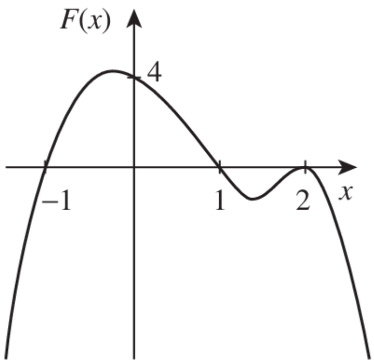
g $y = (2x-3)^2(x+1)^2$



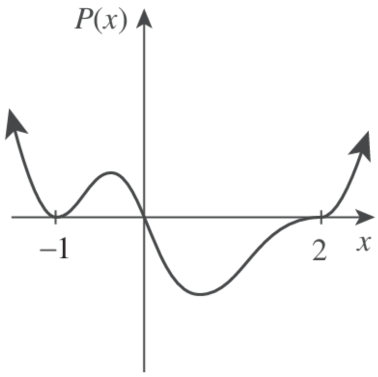
h $y = (1-x)^3(x-3)$



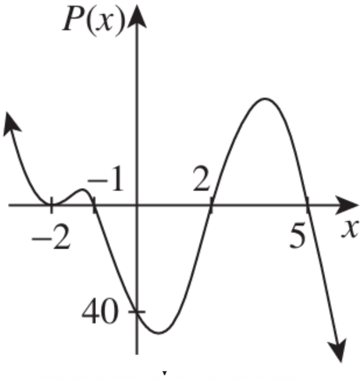
i $y = (2-x)^2(1-x^2)$



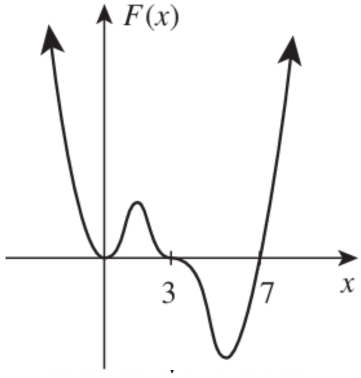
j $y = x(x-2)^3(x+1)^2$



k $y = (x+1)(4-x^2)(x^2-3x-10)$

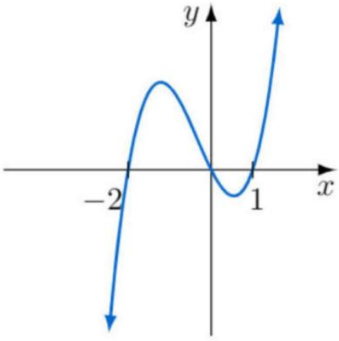


l $y = x^2(x-3)^3(x-7)$



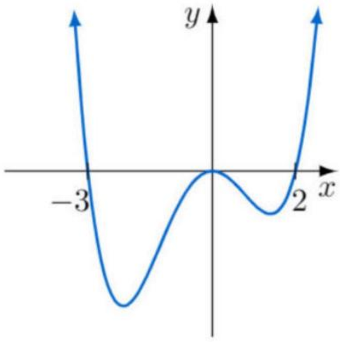
6 The diagrams below have leading coefficients either 1 or -1.
Write the equation of each polynomial.

a



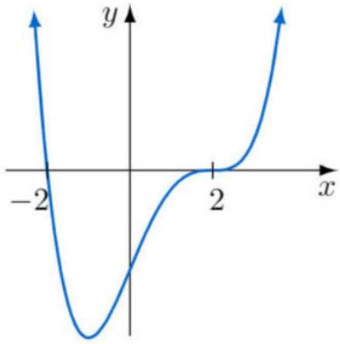
$P(x) = x(x+2)(x-1)$

b



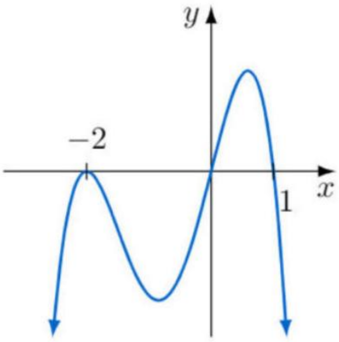
$P(x) = x^2(x+3)(x-2)$

c



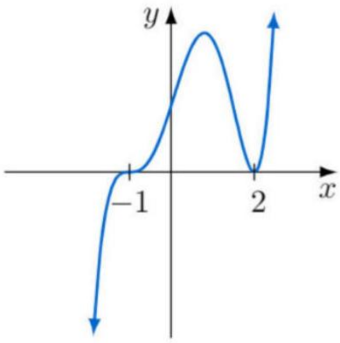
$P(x) = (x+2)(x-2)^2$

d



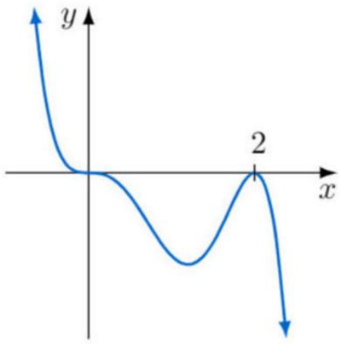
$P(x) = -x(x-1)(x+2)^2$

e



$P(x) = (x-2)^2(x+1)^3$

f



$P(x) = -x^3(x-2)^2$

7 Given $P(x) = 2x + 5$, $Q(x) = x^2 - x - 2$ and $R(x) = x^3 - 9x$, find:

a Zeroes of $P(x)$

$-\frac{5}{2}$

b Zeroes of $Q(x)$

$-1, 2$

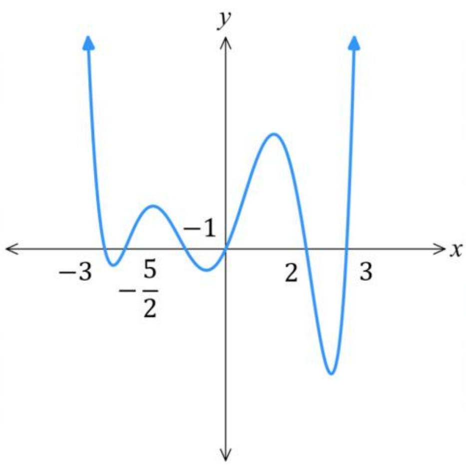
c Zeroes of $R(x)$

$0, 3, -3$

d Hence identify the zeroes of $P(x)Q(x)R(x)$.

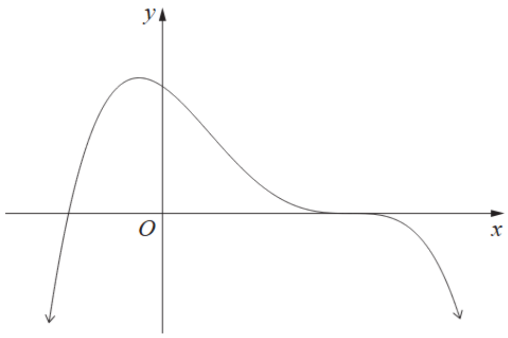
$-\frac{5}{2}, -1, 2, 0, 3, -3$

e Hence sketch $P(x)Q(x)R(x)$.



8 **2021 HSC Advanced Band 4**

The graph of $f(x)$ is shown. Which of the following options could be the equation of this graph?



A. $y=(1-x)(2+x)^3$

B. $y=(x+1)(x-2)^3$

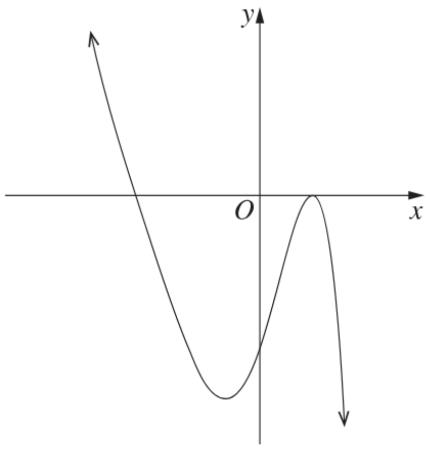
C. $y=(x+1)(2-x)^3$

D. $y=(x-1)(2+x)^3$

C

9 **2023 HSC Advanced Band 4**

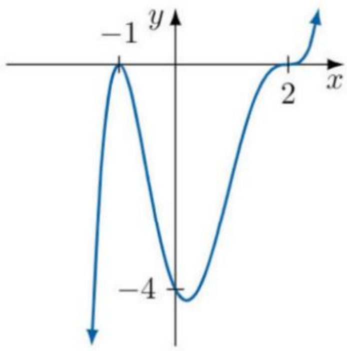
Which row of the table is correct for this cubic curve?



	Equation	*Value of b*	*Value of c*
A.	$y=-(x-b)(x-c)^2$	$b>0$	$c<0$
B.	$y=-(x-b)(x-c)^2$	$b<0$	$c>0$
C.	$y=-x(x-b)(x-c)$	$b>0$	$c<0$
D.	$y=-x(x-b)(x-c)$	$b<0$	$c>0$

B

10 The diagram below shows the graph of a polynomial $y=P(x)$ with roots $x=-1$ and $x=2$.



a Explain why this is a polynomial of the form

$y=a(x+1)^2(x-2)^3$.

The graph shows a double root at $x=-1$ and a triple root at $x=2$

b Find the value of a by using the y-intercept.

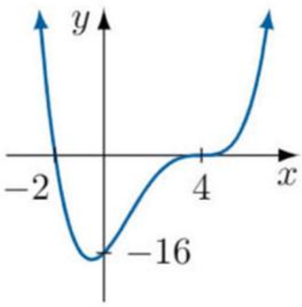
$a=\frac{1}{2}$

c Hence, state the equation of the polynomial.

$P(x)=\frac{1}{2}(x+1)^2(x-2)^3$

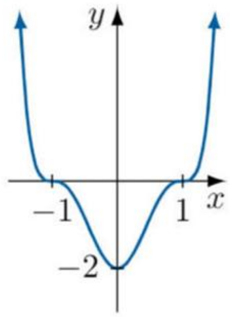
11 Use a similar technique to find the equation of these polynomial graphs.

a



$y = \frac{1}{8}(x-4)^3(x+2)$

b



$y = 2(x-1)^3(x+1)^3$

12 Consider the polynomial $P(x) = x^4 - 5x^2 + 4x + 13$.

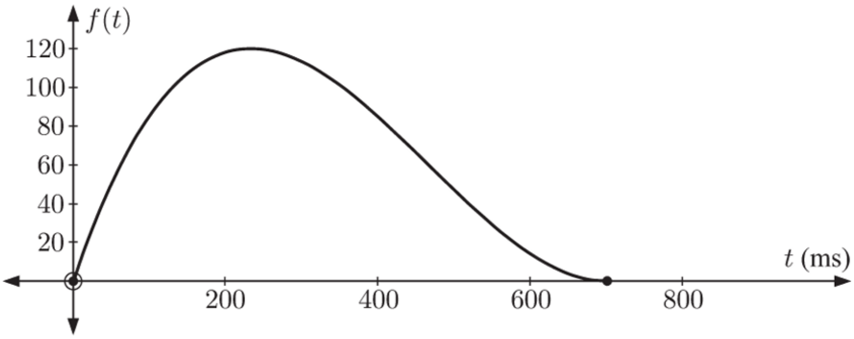
a $P(x)$ can be expressed in the form $(x^2 - a)^2 + (x - b)^2$. Find the values of a and b.

$P(x) = (x^2-3)^2 + (x+2)^2$

b Hence explain why the graph of $P(x)$ has zero x-intercepts.

$P(x) = (x^2-3)^2 + (x+2)^2$ is a sum of two squares, it will always be positive.

13 A scientist is trying to design an ideal crash test barrier with the characteristics shown in the graph below. The independent variable t is time after impact, measured in milliseconds. The dependent variable is the distance the barrier is depressed during the impact, measured in millimetres.



a The equation for this graph has the form $f(t) = kt(t-a)^2$. $0 \le t \le 700$
Use the graph to find the value of a and explain what this represents.

By looking at the zeroes of the graph, $a = 700$. It takes 700 milliseconds for the barrier to bounce back to original after the crash.

b This ideal crash barrier is depressed by 85 mm after 100 milliseconds. Find the value of k that gives this result and hence find the equation of the graph.

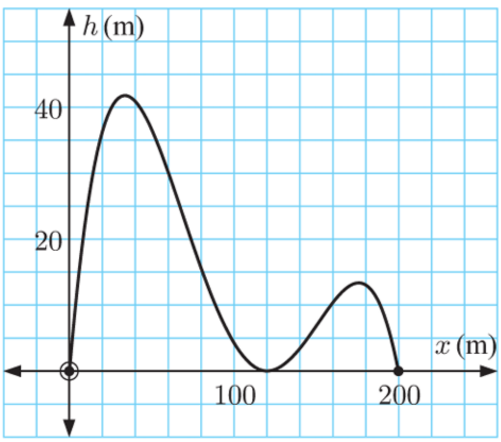
Sub $t = 100, f(t) = 85$ into the equation to solve for k.

$$k = \frac{17}{7200000}$$

$$f(t) = \frac{17}{7200000}t(t-700)^2$$

14 The graph shows the height in metres of a rollercoaster above the ground against the horizontal distance travelled, also in metres. The equation of this graph is in the form

$$h(x) = \frac{-x(x-a)(x-b)^2}{k}$$



a Use the graph to find a and b.

$a = 200, b = 120$

b Given the rollercoaster is 4 m above the ground after travelling 100 m, find the value of k.

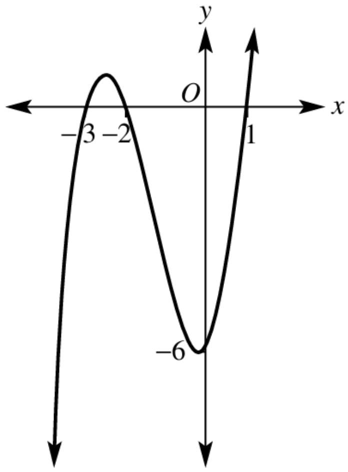
$k = 1\,000\,000$

c Find the height of the rollercoaster above the ground after it has travelled 150 m.

6.75 m

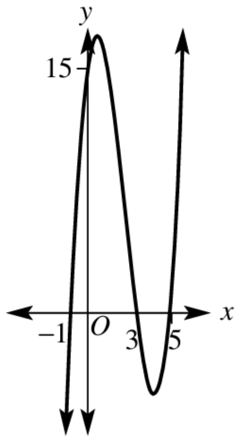
15 Factorise these polynomials using factor theorem and/or long division. Then sketch the graphs.

a $P(x) = x^3 + 4x^2 + x - 6$



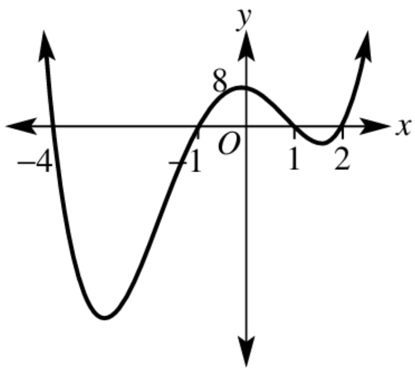
$P(x) = (x - 1)(x + 2)(x + 3)$

b $P(x) = x^3 - 7x^2 + 7x + 15$



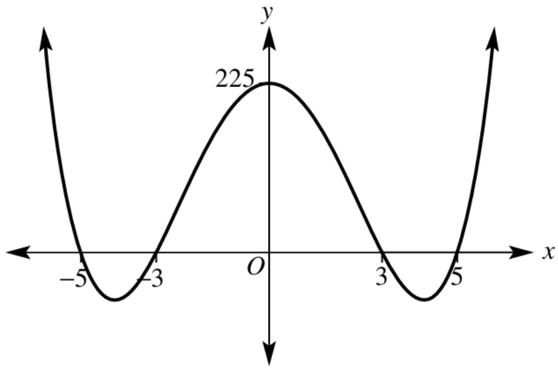
$P(x) = (x - 5)(x - 3)(x + 1)$

c $P(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$



$P(x) = (x - 2)(x - 1)(x + 1)(x + 4)$

d $P(x) = x^4 - 34x^2 + 225$



$P(x) = (x - 5)(x - 3)(x + 3)(x + 5)$