Student Name

Mathematics Stage 5 Paths

NON-LINEAR RELATIONSHIPS C

Book 4 Graph circles and describe their features and transformations

Version: 260207

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CONTENTS

OVERVIEW

SYLLABUS CONTENT

MA5-NLI-P-01 interprets and compares non-linear relationships and their transformations, both algebraically and graphically

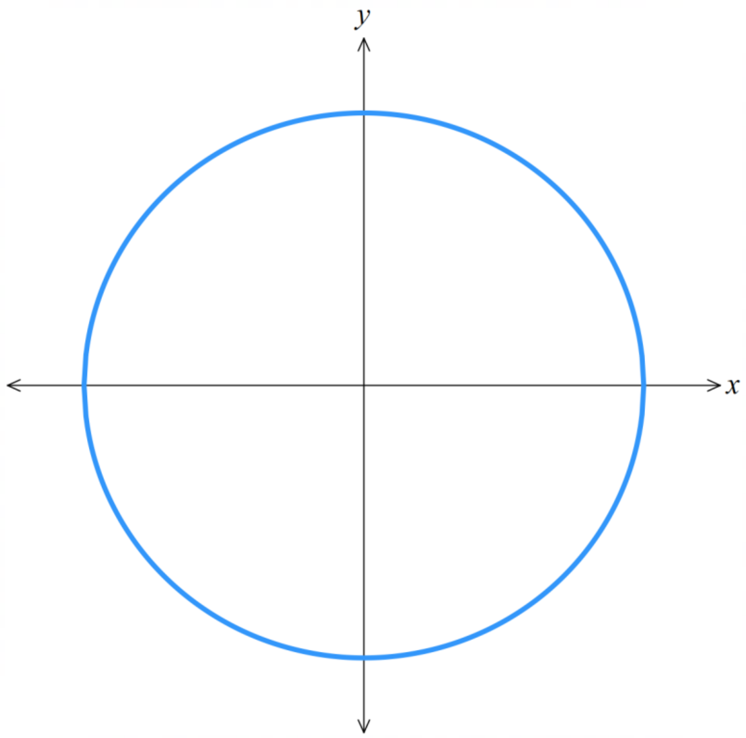
Graph circles and describe their features and transformations

- Derive the equation of a circle $x^2 + y^2 = r^2$ with centre $(0,\ 0)$ and radius r using the distance formula
- Identify and describe equations that represent circles with centre at the origin and radius of the circle r
- Graph circles of the form $x^2 + y^2 = r^2$, where r is the radius of the circle using graphing applications
- Establish the equation of the circle with centre (a, b) and radius r, and graph equations of the form $(x - a)^2 + (y - b)^2 = r^2$
- Find the centre and radius of a circle with the equation in the form $x^2 + y^2 + ax + by + c = 0$ by completing the square

CIRCLES AT THE ORIGIN

Investigation Derivation the equation of a circle centred at the origin.

Consider the following circle with radius r centred at the origin $(0,0)$



1. Choose a point on the circumference. Label this $P(x, y)$

2. Draw a line segment from the centre to the point P.
   What is the length of this line and what is it called? ....

3. Construct a right-angled triangle using the radius as the hypotenuse.
   Label the lengths of the sides.

4. Write an equation relating the three sides of the right-angled triangle.

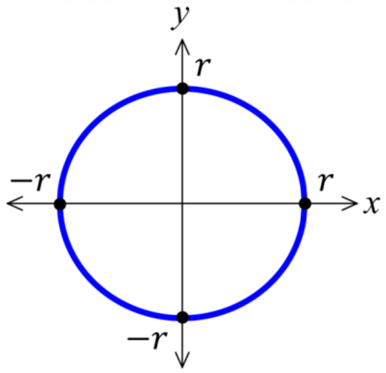
   ....

   This is the equation of a circle.

Equation of a Circle Centred at the Origin

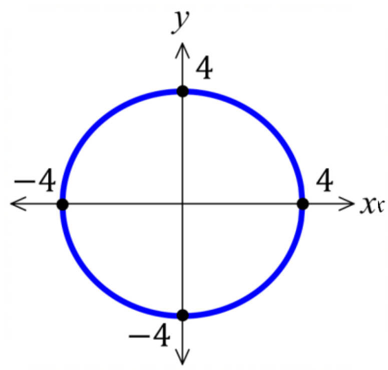
$$x^2 + y^2 = r^2$$

Radius r

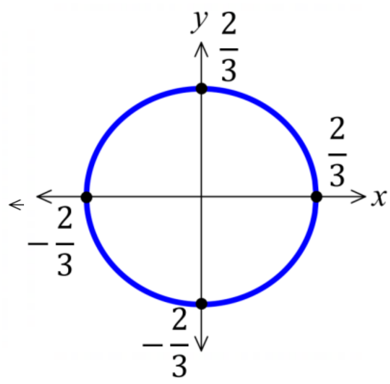


Example Sketch the graph of a circle centred on the origin.

Sketch $x^2 + y^2 = 16$

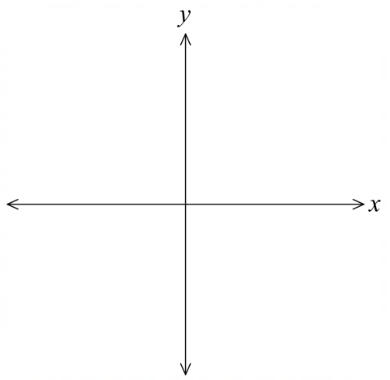


Sketch $x^2 + y^2 = \frac{4}{9}$

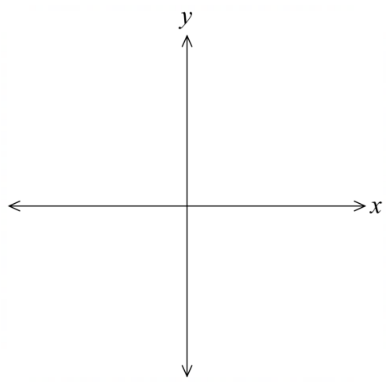


Sketch these circles.

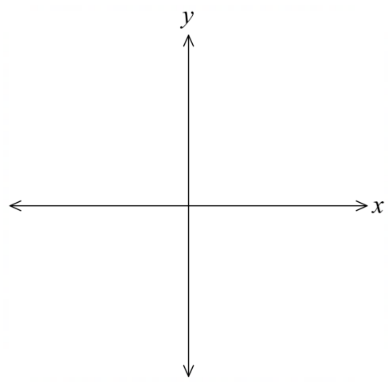
a $x^2 + y^2 = 36$



b $x^2 + y^2 = \frac{1}{4}$



c $x^2 + y^2 = 10$



Key ideas

1. If an equation of a circle is given as $x^2 + y^2 = a$, then the centre is …………… and the radius is ………………

1 Identify the radius of the circles with these equations:

a $x^2 + y^2 = 36$

6

b $x^2 + y^2 = 81$

9

c $x^2 + y^2 = 144$

12

d $x^2 + y^2 = 5$

$\sqrt{5}$

e $x^2 + y^2 = 12$

$2\sqrt{3}$

f $x^2 + y^2 = \frac{1}{4}$

$\frac{1}{2}$

2 Write the equation of a circle with centre at the origin and radius of:

a 4

$x^2 + y^2 = 16$

b 7

$x^2 + y^2 = 49$

c 100

$x^2 + y^2 = 10000$

d $\sqrt{6}$

$x^2 + y^2 = 6$

e $\sqrt{10}$

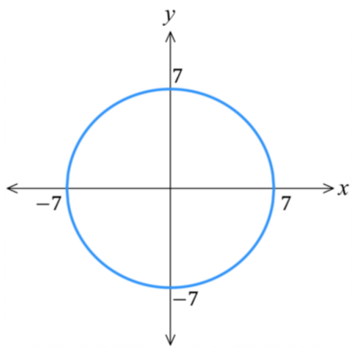
$x^2 + y^2 = 10$

f 1.1

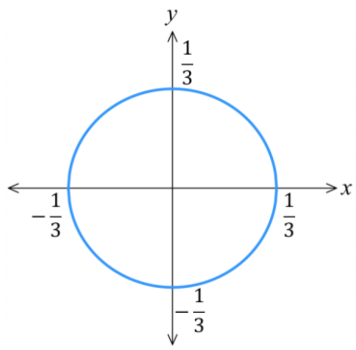
$x^2 + y^2 = 1.21$

3 Sketch these circles showing the centre and all intercepts.

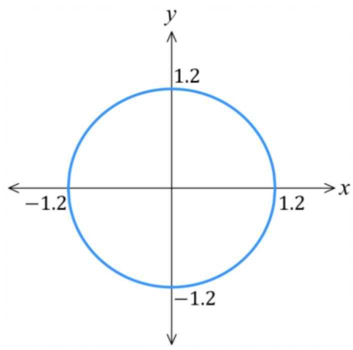
a $x^2 + y^2 = 49$



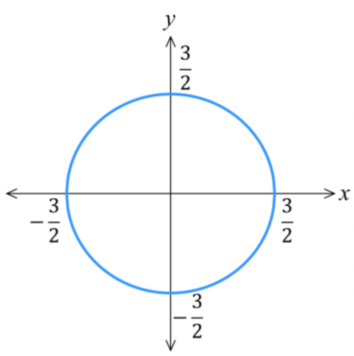
b $x^2 + y^2 = \frac{1}{9}$



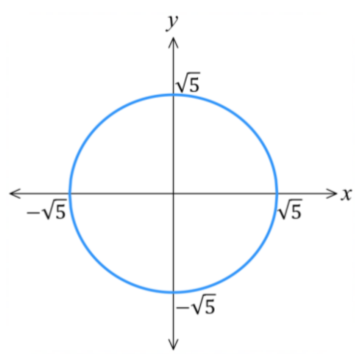
c $x^2 + y^2 = 1.44$



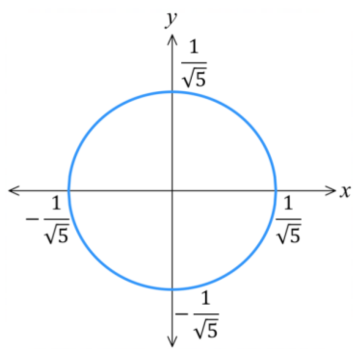
d $x^2 + y^2 = \frac{9}{4}$



e $x^2 + y^2 = 5$

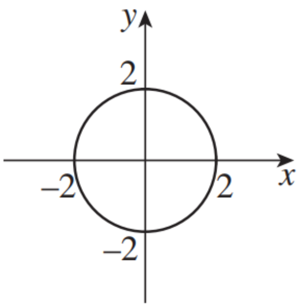


f $x^2 + y^2 = \frac{1}{5}$



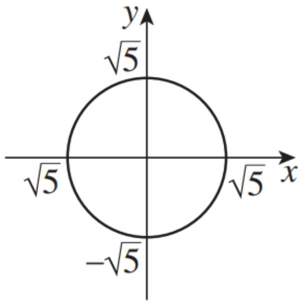
4 Determine the equation of each circle. They are all centred at the origin.

a



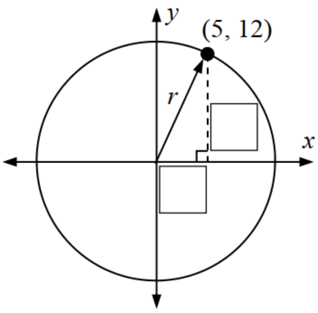
$x^2 + y^2 = 4$

b



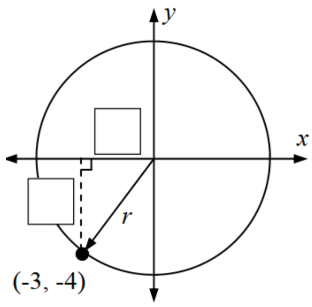
$x^2 + y^2 = 5$

c



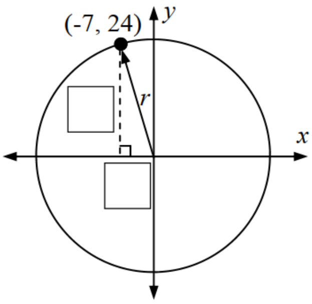
$x^2 + y^2 = 169$

d



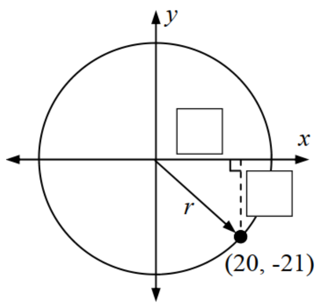
$x^2 + y^2 = 25$

e



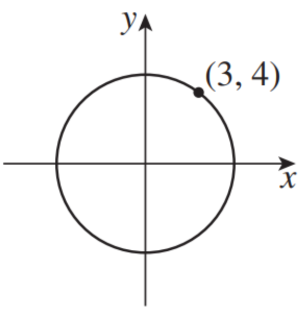
$x^2 + y^2 = 625$

f



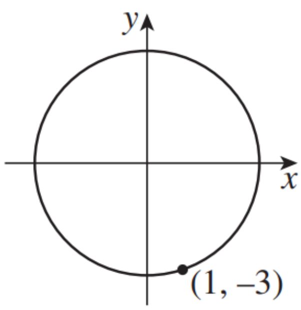
$x^2 + y^2 = 841$

g



$x^2 + y^2 = 25$

h



$x^2 + y^2 = 10$

5 A circle with centre $O(0,0)$ passes through $A(5,12)$.

a What is its radius?

13

b Therefore, what is its equation?

$x^2 + y^2 = 169$

6 Consider the four points $A(1,4), B\left(2,\sqrt{13}\right), C\left(3,2\sqrt{2}\right)$ and $D(4,1)$

a Find the distance of each point from the origin O.

$\sqrt{17}$

b Hence explain why the four points lie on a circle with centre the origin and state the equation of the circle.

Each point is equidistant from the origin, so it lies on a circle centred at the origin.

$x^2 + y^2 = 17$

c What is the radius, diameter, circumference and area of this circle?

$r = \sqrt{17}, d = 2\sqrt{17}, C = 2\pi\sqrt{17}, A = 17\pi$

7 Show, using substitution, that $E(-12,-5)$ lies on the circle centred at the origin with radius 13.

$x^2 + y^2 = 169$

$(-12)^2 + (-5)^2 = 144 + 25 = 169$

8 Consider points $A(-5,0), B(5,0)$, and $C(3,4)$.

a Show that the points all lie on the circle $x^2 + y^2 = 25$.

Using distance formula, distance from origin of all 3 points is 5

b Show that $AC \perp BC$.

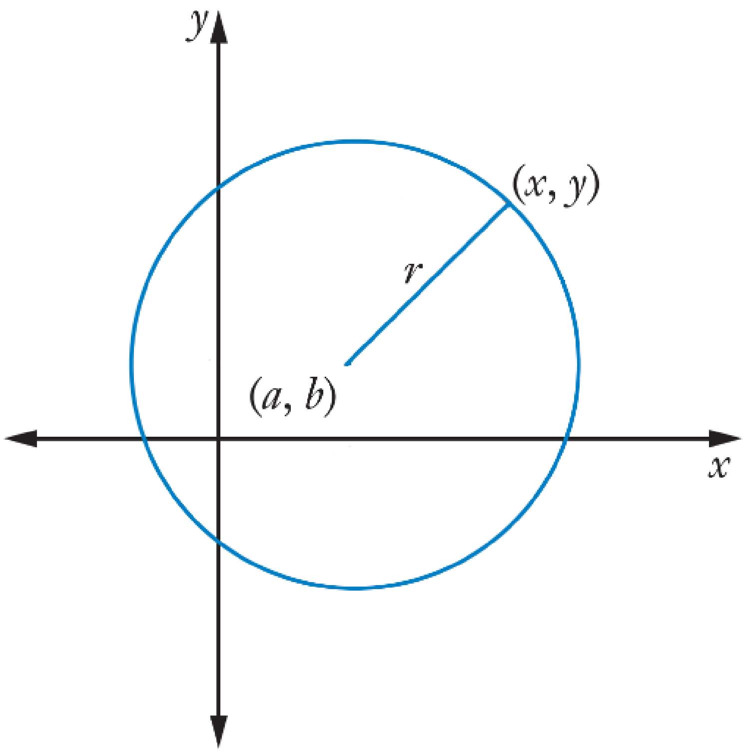
$m_{AC} = \frac{1}{2}, m_{BC} = -2$

Since the gradients of AC and BC are negative reciprocals, they are perpendicular.

SKETCHING CIRCLES NOT AT THE ORIGIN

Investigation: Derivation of general circle equation.

Consider the following circle with radius r centred at (a, b)



Construct a right-angled triangle using the radius as the hypotenuse.
Label the lengths of the sides.

Write an equation relating the three sides of the right-angled triangle.
This is the general equation of a circle.

……………………………………………………………………………………………………

General Equation of a Circle

Centre (a, b)

$$(x-a)^2+(y-b)^2=r^2$$

Radius r

1 Choose the word *left, right, up* or *down* to complete each sentence.

a Compared to $x^2+y^2=r^2$, $(x-1)^2+y^2=r^2$ is translated …………… by 1 unit.

right

b Compared to $x^2+y^2=r^2$, $(x+3)^2+y^2=r^2$ is translated ……………………………

Left by 3 units

c Compared to $x^2+y^2=r^2$, $x^2+(y+3)^2=r^2$ is translated ……………………………

Down by 3 units

d Compared to $x^2+y^2=r^2$, $x^2+(y-2)^2=r^2$ is translated ……………………………

Up by 2 units

e Compared to $x^2+y^2=r^2$, $(x+2)^2+(y-1)^2=r^2$ is translated

………………………………………………………………………

Left by 2, up by 1

f Compared to $x^2+y^2=r^2$, $(x+5)^2+(y+3)^2=r^2$ is translated

………………………………………………………………………

Left by 5, down by 3

g Compared to $x^2+y^2=r^2$, $(x-4)^2+(y+2)^2=r^2$ is translated

………………………………………………………………………

Right by 4, down by 2

2 Write the equation of the circle with:

a centre $(-1,5)$, radius 4

$(x+1)^2+(y-5)^2=16$

b centre $(3,-5)$, radius $\sqrt{7}$

$(x-3)^2+(y+5)^2=7$

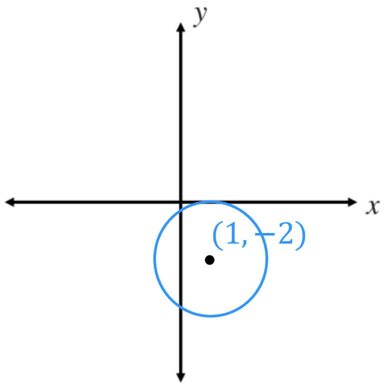
c $(8,11)$, radius $2\sqrt{5}$

$(x-8)^2+(y-11)^2=20$

Example Sketch the graph of a circle not centred at origin.

Sketch $(x-1)^2+(y+2)^2=4$

Circle with radius 2, centred at $(1,-2)$



How did we know the circle passes the y-axis but does not pass the x-axis?

The distance from the centre to the y-axis is 1, which is ………… than the radius.

The distance from the centre to the x-axis is ……, which is ………… to the radius.

Find the intercepts.

x-intercept:

$(x-1)^2+(0+2)^2=4$

$(1,0)$

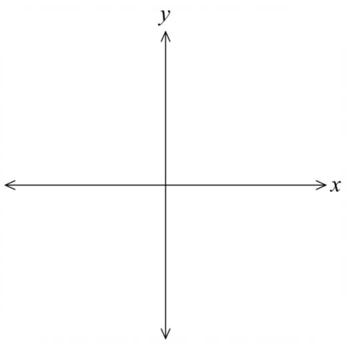
y-intercepts:

$(0-1)^2+(y+2)^2=4$

$(0,-2+\sqrt{3})$ and $(0,-2-\sqrt{3})$

Sketch these circles, then find the intercepts (if any).

a $(x-1)^2+y^2=1$



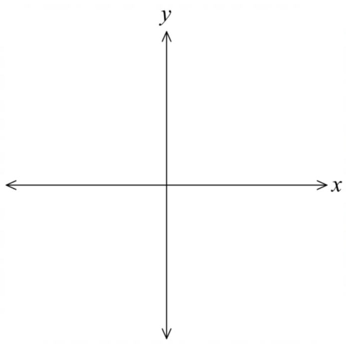
x-intercepts:

$(0,0)$ and $(2,0)$

y-intercepts:

$(0,0)$

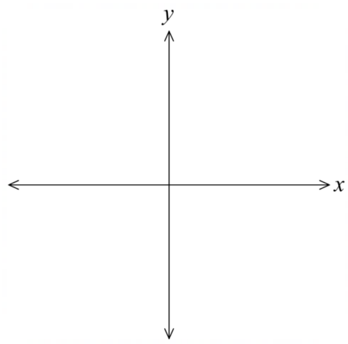
b $x^2+(y+1)^2=4$



$(-\sqrt{3},0)$ and $(\sqrt{3},0)$

$(0,1)$ and $(0,-3)$

c $(x-2)^2+(y+3)^2=9$

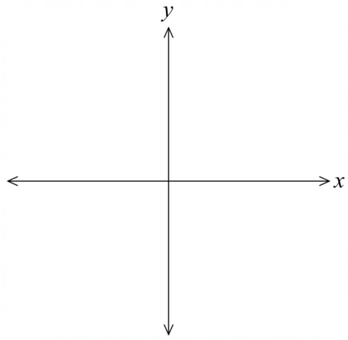


$(2,0)$

$(0,-3+\sqrt{5})$ and $(0,-3-\sqrt{5})$

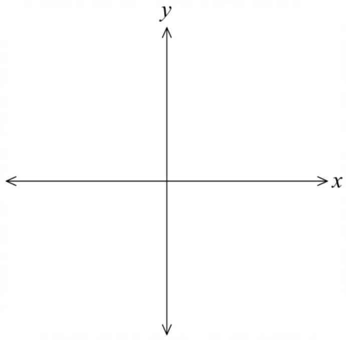
1 Sketch the following circles.
Label the coordinates of the centre and intercepts, if any. Check your sketch using DESMOS.

a $(x-1)^2+y^2=1$



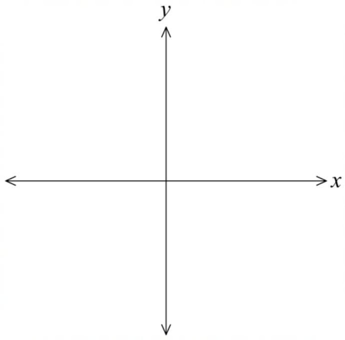
Centre: (1,0)
x-intercepts:$x=0,2$
y-intercepts:$y=0$

b $(x+1)^2+y^2=9$



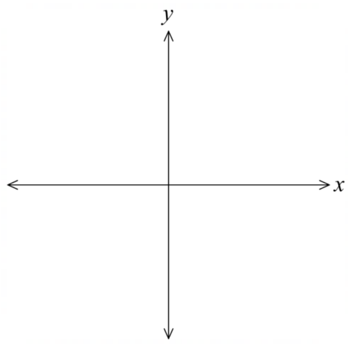
Centre: (−1,0)
x-intercepts:$x=-4,2$
y-intercepts:$y=\pm 2\sqrt{2}$

c $x^2+(y-4)^2=36$



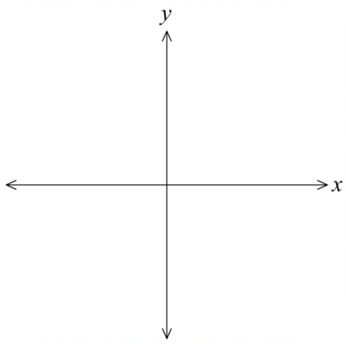
Centre: (0,4)
x-intercepts:$x=3$
y-intercepts: none

d $(x-3)^2+(y+1)^2=1$



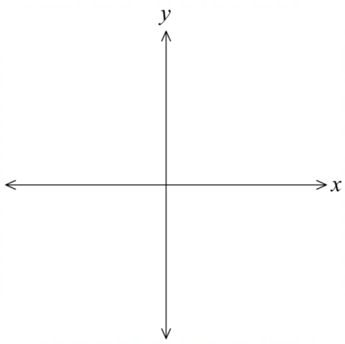
Centre: (3,−1)
x-intercepts:$x=3$
y-intercepts: none

e $(x+2)^2+(y-3)^2=4$



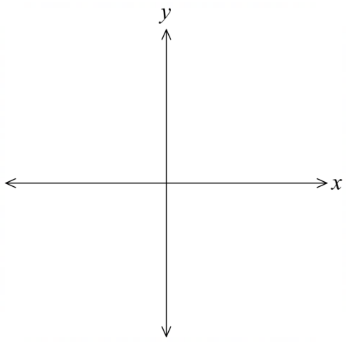
Centre: (−2,3)
x-intercepts: none
y-intercepts:3

f $(x-1)^2+(y+3)^2=25$



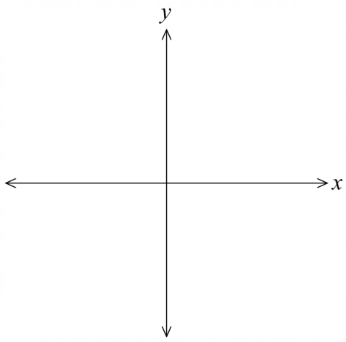
Centre: (1,−3)
x-intercepts:$x=-3,5$
y-intercepts:$y=-3\pm 2\sqrt{6}$

g $(x+3)^2+(y-2)^2=25$



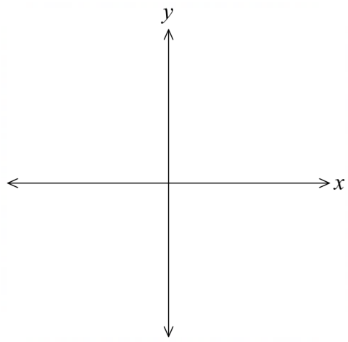
Centre: $(-3,2)$
x-intercepts: $x=-3\pm\sqrt{21}$
y-intercepts: $y=-2,6$

h $(x+2)^2+(y-1)^2=9$



Centre: $(-2,1)$
x-intercepts: $x=-2\pm2\sqrt{2}$
y-intercepts: $y=1\pm\sqrt{5}$

i $(x-2)^2+(y-5)^2=64$



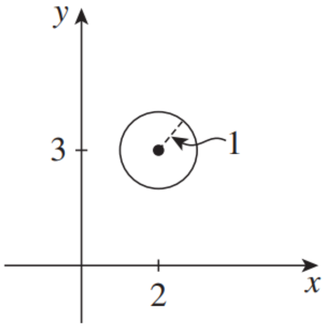
Centre: $(2,5)$
x-intercepts: $x=2\pm\sqrt{39}$
y-intercepts: $y=5\pm2\sqrt{15}$

2 A circle with centre $B(4,5)$ passes through the origin. What is its equation?

$(x-4)^2+(y-5)^2=41$

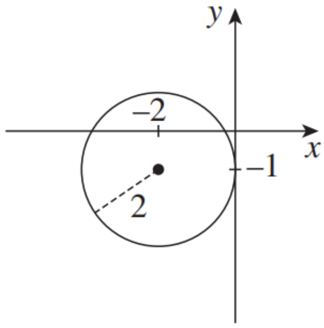
3 Determine the equation of each circle.

a



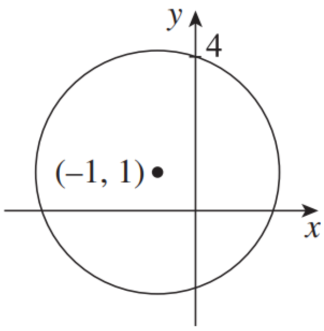
$(x-2)^2+(y-3)^2=1$

b



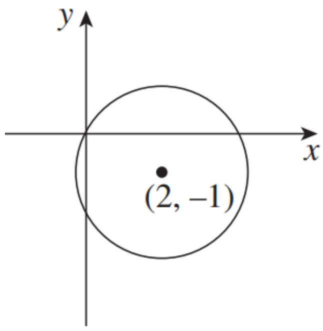
$(x+2)^2+(y+1)^2=4$

c



$(x+1)^2+(y-1)^2=10$

d



$(x-2)^2+(y+1)^2=5$

4 Find the equation of a circle with centre $(5,-2)$ passing through $(-1,1)$.

Distance between the centre and the point is $\sqrt{45}$, so the radius is $\sqrt{45}$.

$(x-5)^2+(y+2)^2=45$

5 A circle has centre $(4,-3)$ and one of the x-intercepts is at $(-\sqrt{7}+4,0)$.
Determine the equation of the circle.

Distance between the centre and the intercept is 4, so the radius is 4.

$(x-4)^2+(y+3)^2=16$

6 The interval joining the points $A(3,-1)$ and $B(-7,5)$ is a diameter of a circle.

a Find the midpoint of AB.

$(-2,2)$

b Explain why the midpoint of AB is the centre of the circle.

The diameter connects two points on the circumference, passing through the centre.
Therefore, the midpoint of the diameter is the centre of the circle.

c Find the length of the radius.

$\sqrt{34}$

d Hence state the equation of the circle.

$(x+2)^2+(y-2)^2=34$

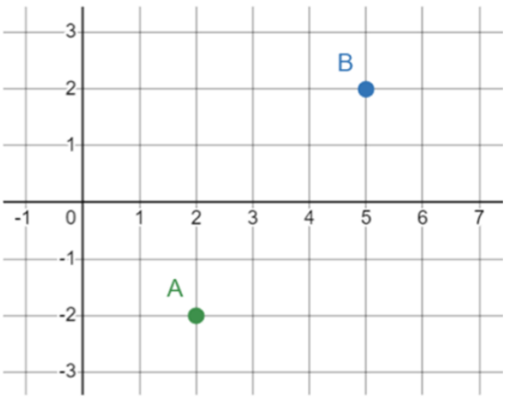
7 Find the centre of the circle with diameter CD, where $C = (2, 1)$ and $D = (8, -7)$.

$(5, -3)$

8 Find the equation of a circle with diameter KL, where $K(5,7)$ and $L(-9,3)$

$(x + 2)^2 + (y - 5)^2 = 53$

9 Find the possible equations of a circle with radius AB.



Circle with centre at A:

$(x - 2)^2 + (y + 2)^2 = 25$

Circle with centre at B:

$(x - 5)^2 + (y - 2)^2 = 25$

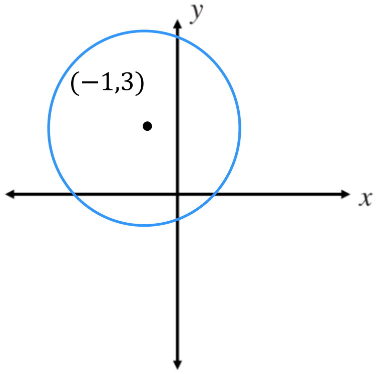
COMPLETING THE SQUARE TO SKETCH A CIRCLE

- To sketch $x^2 + y^2 + ax + by + c = 0$,
- This is a **circle**; **complete the square** for the x-terms and y-terms.

Example Sketch the graph of a circle by CTS.

Sketch the graph of the circle.

$$x^2 + 2x + y^2 - 6y - 6 = 0$$
$$x^2 + 2x + \mathbf{1} - \mathbf{1} + y^2 - 6y + \mathbf{9} - \mathbf{9} - 6 = 0$$
$$(x^2 + 2x + 1) - 1 + (y^2 - 6y + 9) - 9 - 6 = 0$$
$$(x + 1)^2 + (y - 3)^2 - 16 = 0$$
$$(x + 1)^2 + (y - 3)^2 = 16$$



Sketch these circles by completing the square first.

a $x^2 - 4x + y^2 = 0$

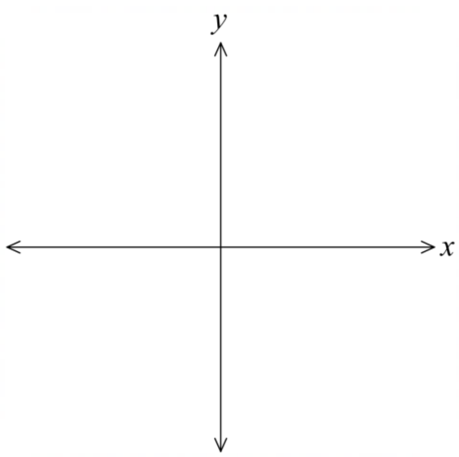
$x^2 - 4x + 4 - 4 + y^2 = 0$

$(x^2 - 4x + 4) - 4 + y^2 = 0$

....

....

$(x - 2)^2 + y^2 = 4$



b $x^2 + 6x + y^2 - 8y = 0$

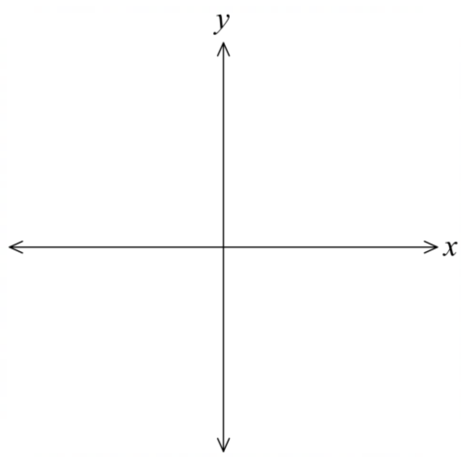
$x^2 + 6x + 9 - 9 + y^2 - 8y + 16 - 16 = 0$

....

....

....

$(x - 3)^2 + (y - 4)^2 = 25$



1 Determine the centre and radius of these circles by completing the square, if necessary.

a $(x+1)^2+y^2=4$

Centre: $(-1,0)$ Radius: 2

b $(x-1)^2+(y-2)^2=1$

Centre: $(1,2)$ Radius: 1

c $x^2-10x+y^2=-16$

Centre: $(5,0)$ Radius: 3

d $x^2+y^2+14x=33$

Centre: $(-7,0)$ Radius: $\sqrt{82}$

e $x^2-2x+y^2-4y-4=0$

Centre: $(1,2)$ Radius: 3

f $x^2+6x+y^2-8y=0$

Centre: $(-3,4)$ Radius: 5

g $x^2-10x+y^2+8y+32=0$

Centre: $(5,-4)$ Radius: 3

h $x^2+14x+14+y^2-2y=0$

Centre: $(-7,1)$ Radius: 6

i $x^2+4x+y^2-2y+1=0$

Centre: $(-2,1)$ Radius: 2

j $x^2-6x+y^2-4y-3=0$

Centre: $(3,2)$ Radius: 4

2

a Complete the square for $x^2+4x+y^2-6y+15=0$.

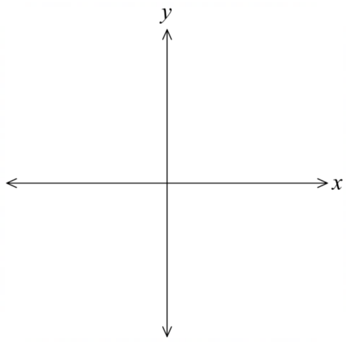
$(x+2)^2+(y-3)^2=-2$

b Explain why this is not an equation of a circle.

The radius cannot be negative.

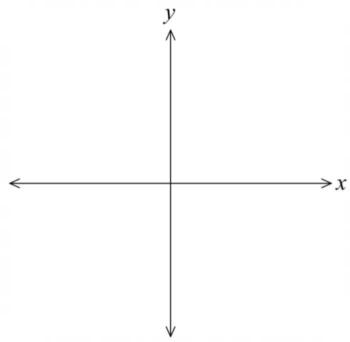
3 Complete the square to sketch these circles. Show the centre, radius, and intercepts.

a $x^2 - 2x + y^2 = 0$



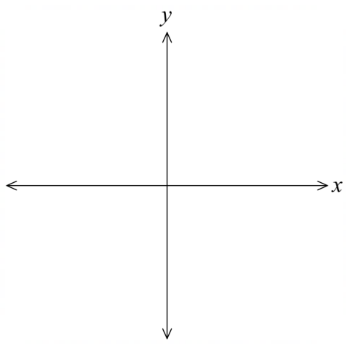
Centre: (1,0)
Radius: 1
x-intercepts: $x = 0, 2$
y-intercepts: $y = 0$

b $x^2 + 6x + y^2 = 0$



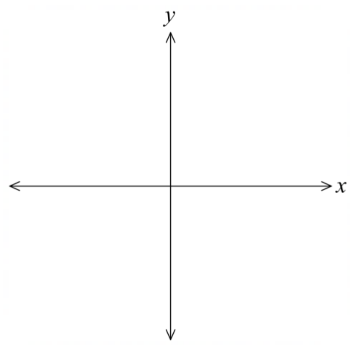
Centre: (–3,0)
Radius: 3
x-intercepts: $x = -6, 0$
y-intercepts: $y = 0$

c $x^2 + 4x + y^2 - 2y + 1 = 0$



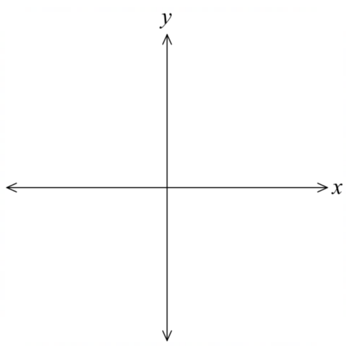
Centre: (–2,1)
Radius: 2
x-intercepts: $x = -2 \pm \sqrt{3}$
y-intercepts: $y = 1$

d $x^2 + 8x + y^2 + 10y + 5 = 0$



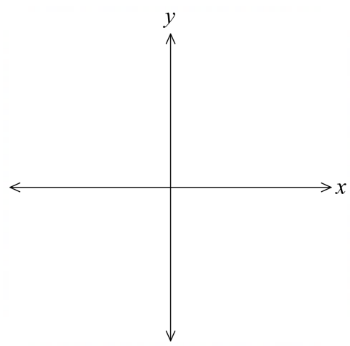
Centre: (–4,–5)
Radius: 6
x-intercepts: $x = -4 \pm \sqrt{11}$
y-intercepts: $y = -5 \pm 2\sqrt{5}$

e $x^2 - 6x + y^2 - 4y - 3 = 0$



Centre: (3,2)
Radius: 4
x-intercepts: $x = 3 \pm 2\sqrt{3}$
y-intercepts: $y = 2 \pm \sqrt{7}$

f $x^2 - 2x + y^2 + 6y - 5 = 0$



Centre: (1,–3)
Radius: $\sqrt{15}$
x-intercepts: $x = 1 \pm \sqrt{6}$
y-intercepts: $y = -3 \pm \sqrt{14}$

4 For what values of k would the graph of $x^2 + 6x + y^2 + k = 0$ not exist?

By CTS, $(x+3)^2 + y^2 = 9 - k$

For circle to not exist, $9 - k < 0$

$\therefore k > 9$

5 A naval ship is at coordinates P and equipped with a radar scanner that can detect objects inside the circle $x^2 + y^2 - 80x - 40y = 500$, where radius is measured in kilometres.

a Find the coordinates of P.

By CTS, $(x-40)^2 + (y-20)^2 = 2500$

P is the centre of the circle, at $(40,20)$

b Another ship is at $Q(60, -20)$ and can detect objects within a radius of 40 km.

i Will ship P be able to detect Q?

Distance between PQ is $\sqrt{2000} \approx 44.7$ km. P's radar has a range of 40 km, so no.

ii Will ship Q be able to detect P?

Since Q has a radar range of 50 km, yes

6 Consider the points $A(1,3)$, $B(3,9)$ and let $P(x,y)$ be a point.

a Find an expression for the gradients of PA and PB.

$m_{PA} = \frac{y-3}{x-1}$, $m_{PB} = \frac{y-9}{x-3}$

b Suppose $\angle APB = 90°$. Show that $x^2 - 4x + y^2 - 12y + 30 = 0$.

$$m_{PA} \times m_{PB} = -1$$
$$\frac{y-3}{x-1} \times \frac{y-9}{x-3} = -1$$
$$\frac{(y-3)(y-9)}{(x-1)(x-3)} = -1$$
$$(y-3)(y-9) = -(x-1)(x-3)$$
$$y^2 - 12y + 27 = -x^2 + 4x - 3$$
$$x^2 - 4x + y^2 - 12y + 30 = 0$$

c By completing the square, show that this is the equation of a circle centred at $(2,6)$ with radius $\sqrt{10}$.

d How is the point P related to the circle? A sketch using DESMOS may help.
https://www.desmos.com/geometry/opfcwyddok

We have proved that if we draw any point P and connect it to both A and B such that it forms a right angle at $\angle APB$, the point P must lie on a circle centred at the midpoint of AB with radius $\sqrt{10}$.

This is related to the circle theorem "The angle at the circumference in a semicircle is 90°"