

Student Name

Mathematics Stage 5 Paths

FUNCTIONS AND OTHER GRAPHS

Book 2 Find the domain and range of a function and graph functions.

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OVERVIEW

SYLLABUS CONTENT

MA5-FNC-P-01 uses function notation to describe and graph functions of one variable and graphs inequalities in one and 2 variables

Find the domain and range of a function and graph functions

- Define the domain as the set of all allowable values of x
- Define the range as the set of possible y -values as x varies over the domain of the function
- Determine and describe the domain and range for a variety of functions
- Use graphing applications to graph and compare functions of the form $y = f(x)$, $y = f(x) + c$, $y = f(x - b)$, $y = kf(x)$ and $y = f(ax)$, and describe their transformations

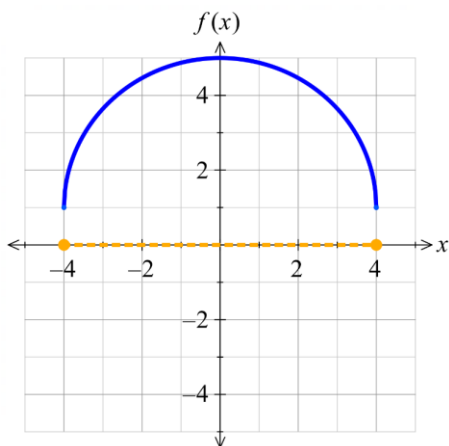
REVIEW OF PRIOR KNOWLEDGE

DOMAIN AND RANGE

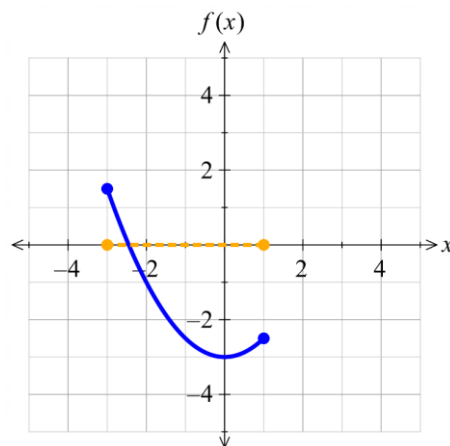
Investigation Domain

The domain is the allowable inputs of a function.

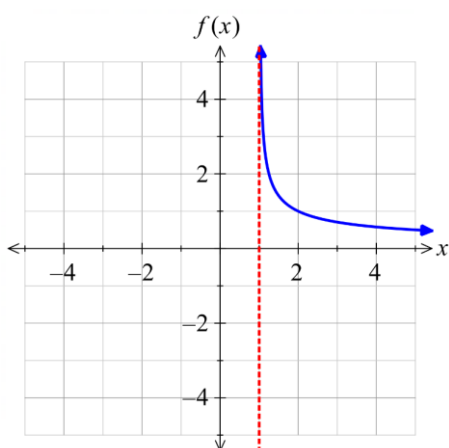
We can specify a domain with an interval, and we can visualise it as a shadow on the x -axis.



The domain is $-4 \leq x \leq 4$



The domain is $..... \leq x \leq$

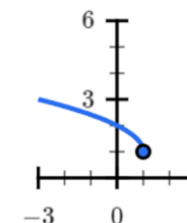
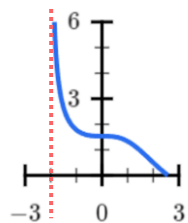
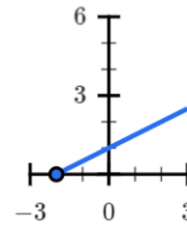
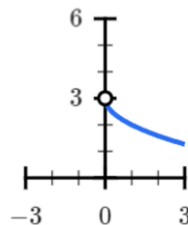


The domain is: (choose one)

- ☐ $x \geq 1$
- ☐ $x > 1$
- ☐ x can be any real number

Match the domain to the function.

$x > -2$ $x \geq -2$ $x \leq 1$ $x > 0$



Investigation Interval notation.

There are two ways of writing the same interval:

$$-1 \leq x < 3$$

$$[-1, 3)$$

What is true about the [and) symbols?

- ☐ [and] are used to include a value, just like $\leq$ and $\geq$
- ☐ (and) are used to exclude a value, just like $<$ and $>$
- ☐ Both above statements are true.

What does $[2, 4]$ mean?

- ☐ $2 \leq x \leq 4$
- ☐ $2 < x < 4$
- ☐ $2 \leq x < 4$

What does $[2, 4)$ mean?

- ☐ $2 \leq x \leq 4$
- ☐ $2 < x < 4$
- ☐ $2 \leq x < 4$

Match the interval to the inequality.

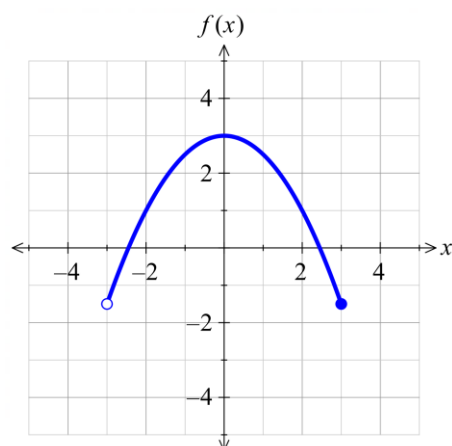
$$-3 \leq x < 1 \quad (1, 3]$$

$$-3 < x < 1 \quad [1, 3]$$

$$1 \leq x \leq 3 \quad (-3, 1)$$

$$1 < x \leq 3 \quad [-3, 1)$$

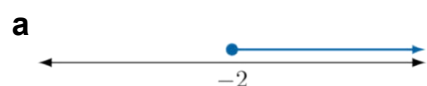
What is the domain, using interval notation?



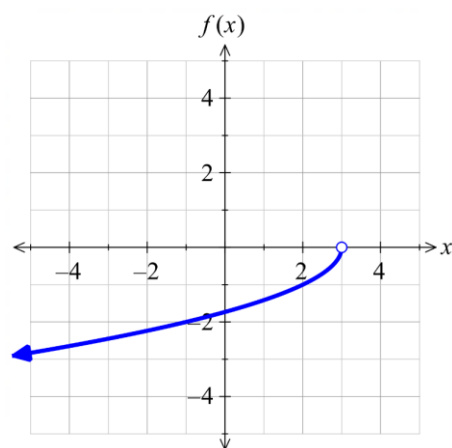
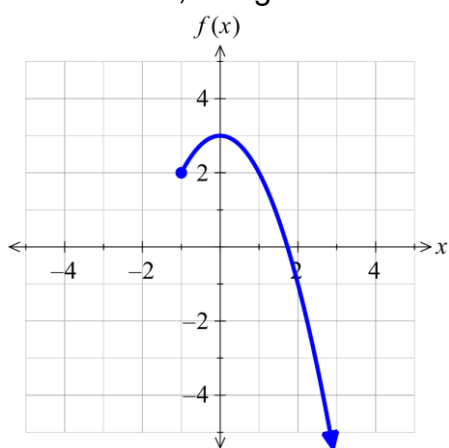
How would you represent $x \geq 0$ in interval notation?

- ☐ $[0, \infty)$ since ∞ can't be an included value.
- ☐ $[0, \infty]$ since ∞ can be an included value.

Write these inequalities using interval notation.



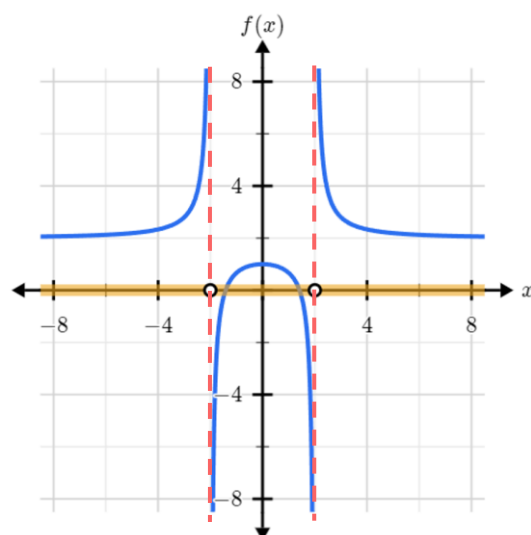
What is the domain, using interval notation?



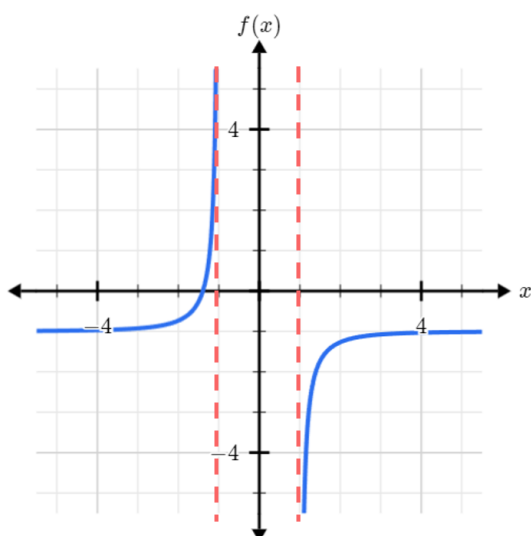
We can combine separate intervals of a domain with the union symbol $\cup$.

What is the domain of the graphed function?

- ☐ $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
- ☐ $(-\infty, -2] \cup [-2, 2] \cup [2, \infty)$
- ☐ $[-\infty, -2] \cup (-2, 2) \cup (2, \infty]$



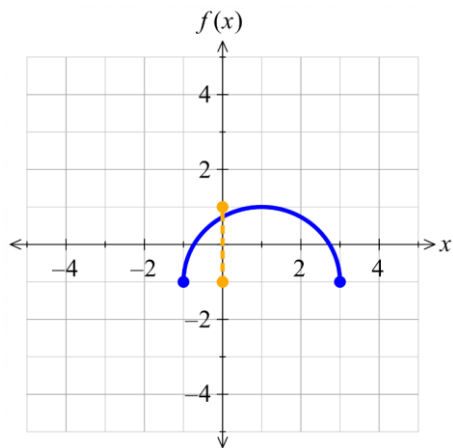
What is the domain of the graphed function?



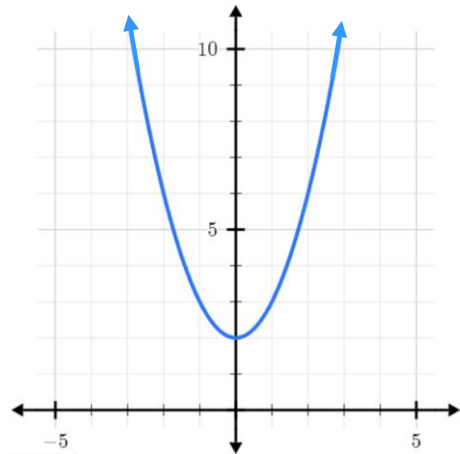
Investigation Range

The range is the possible outputs of a function.

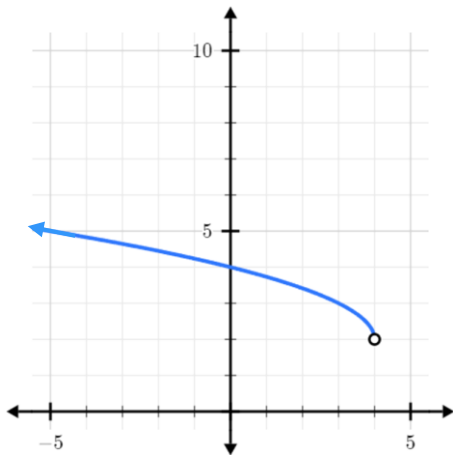
We can specify a range with an interval, and we can visualise it as a shadow on the y-axis.



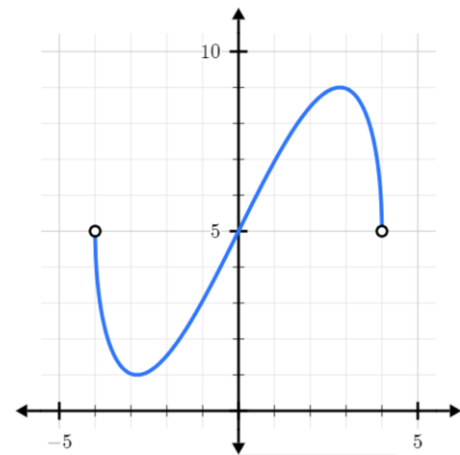
The range is $[-1, 1]$



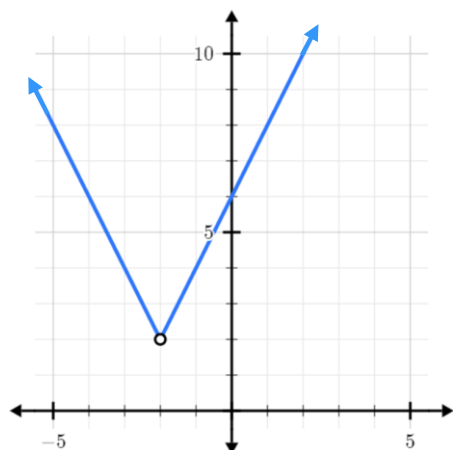
The range is



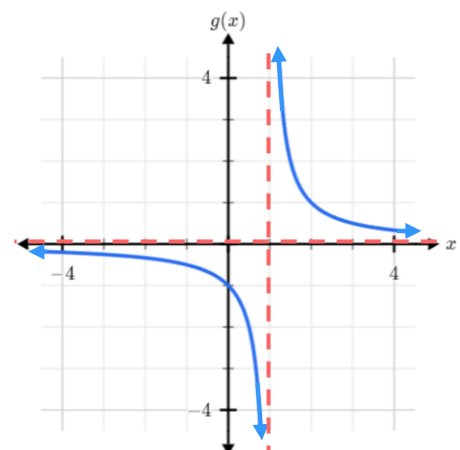
The range is



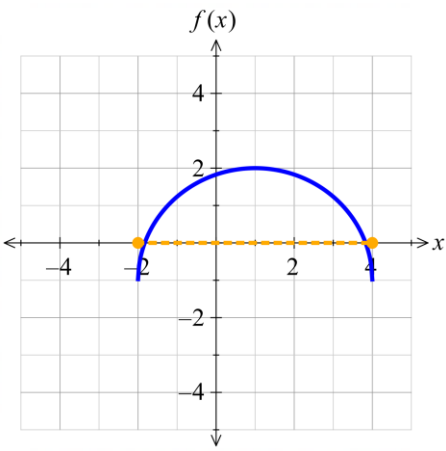
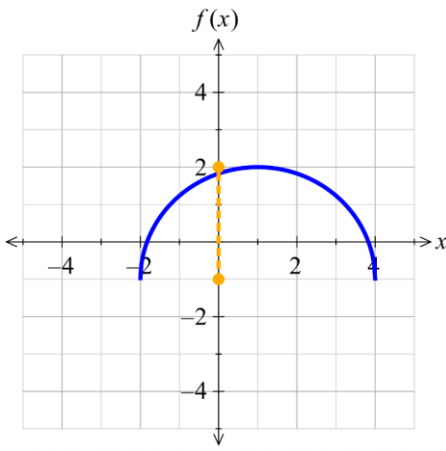
The range is



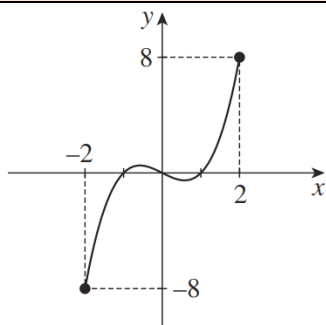
The range is



The range is

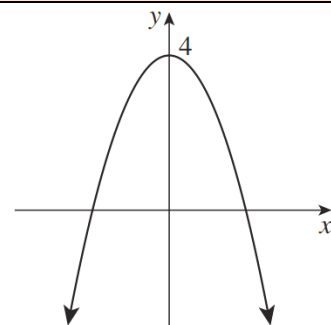
Domain	Range
The domain of a function is the set of all allowable x-values. Visualise it as a shadow on the x-axis. 	The range of a function is the set of all possible y-values. Visualise it as a shadow on the y-axis. 

Example Identify domain and range from a graph.



Domain: $-2 \leq x \leq 2$ $[-2, 2]$

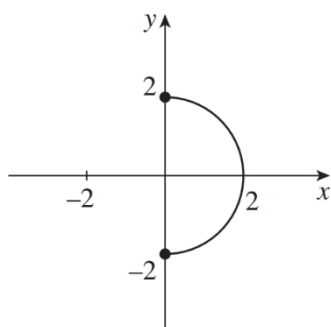
Range: $-8 \leq y \leq 8$ $[-8, 8]$



Domain: all real x $x \in \mathbb{R}$

Range: $y \leq 4$ $(-\infty, 4]$

a



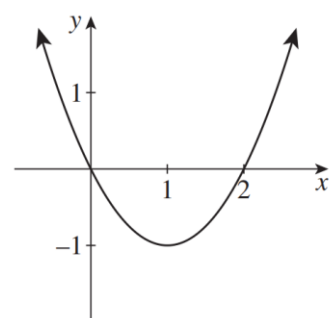
Domain:

$x \geq 0$ $[0, 2]$

Range:

$-2 \leq y \leq 2$ $[-2, 2]$

b



Domain:

all real x $x \in \mathbb{R}$

Range:

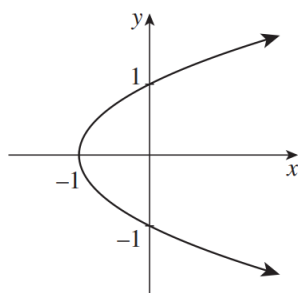
$y \geq -1$ $[-1, \infty)$

1 Identify the domain and range of these graphs.

a

Domain:

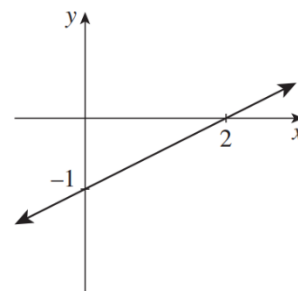
Range:



b

Domain:

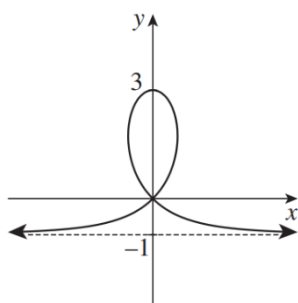
Range:



c

Domain:

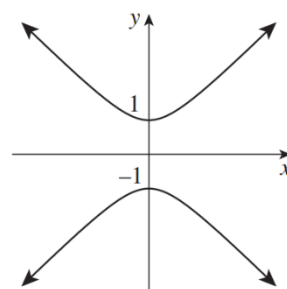
Range:



d

Domain:

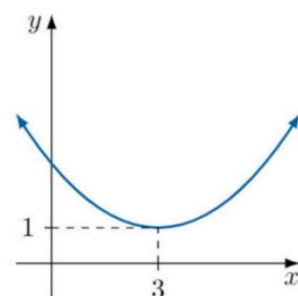
Range:



e

Domain:

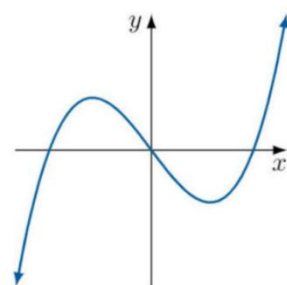
Range:



f

Domain:

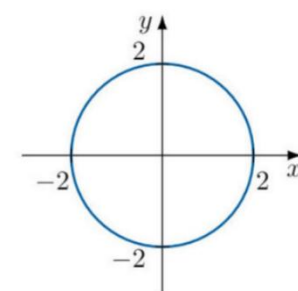
Range:



g

Domain:

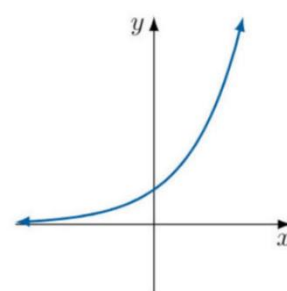
Range:



h

Domain:

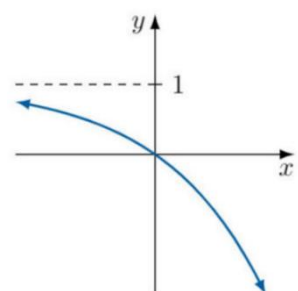
Range:



i

Domain:

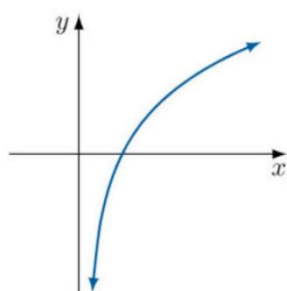
Range:



j

Domain:

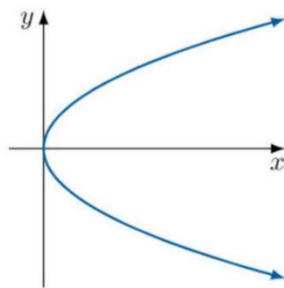
Range:



k

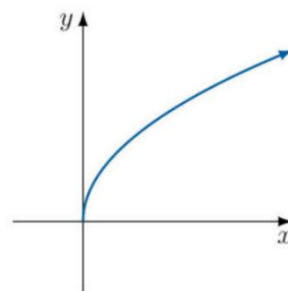
Domain:

Range:

**l**

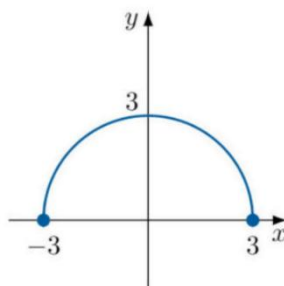
Domain:

Range:

**m**

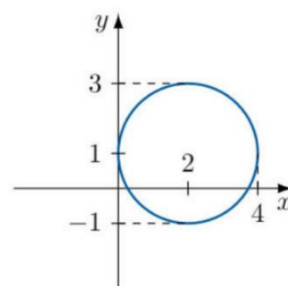
Domain:

Range:

**n**

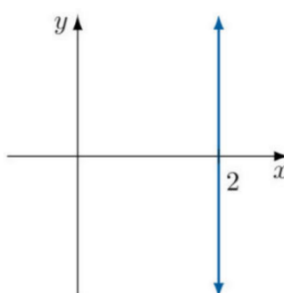
Domain:

Range:

**o**

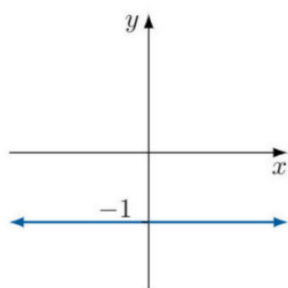
Domain:

Range:

**p**

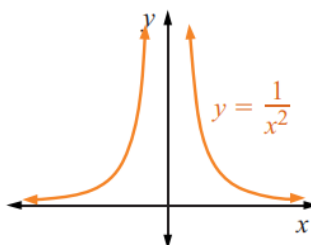
Domain:

Range:

**q**

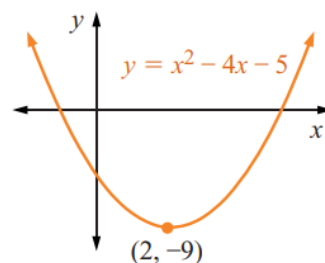
Domain:

Range:

**r**

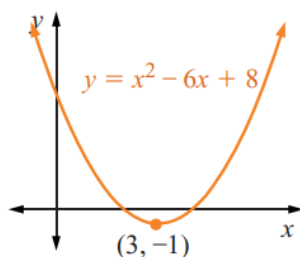
Domain:

Range:

**s**

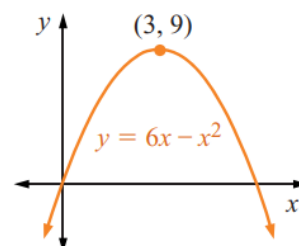
Domain:

Range:

**t**

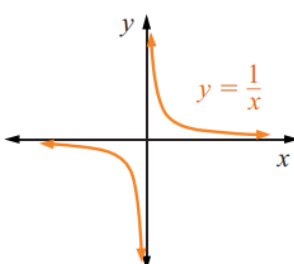
Domain:

Range:

**u**

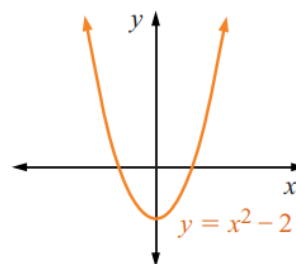
Domain:

Range:

**v**

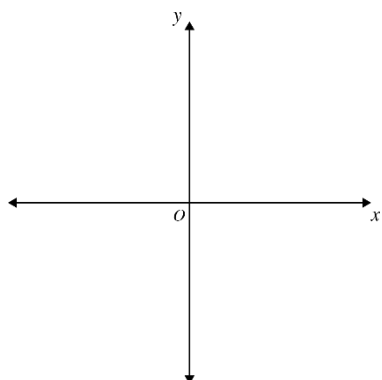
Domain:

Range:

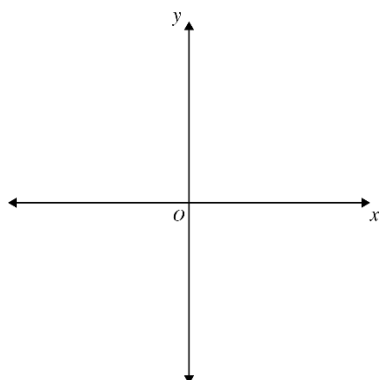


2 Use a sketch to find the range of $y = x^2$, given these domain restrictions.

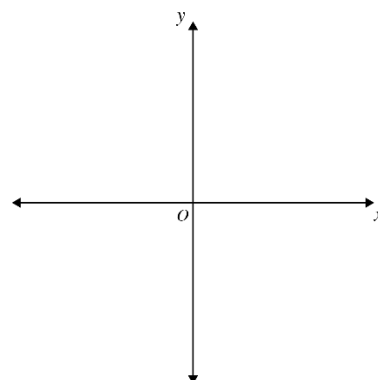
a $x \geq 0$



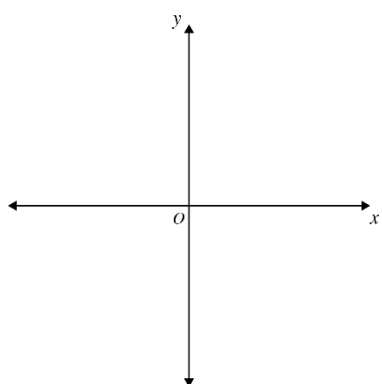
b $x > 0$



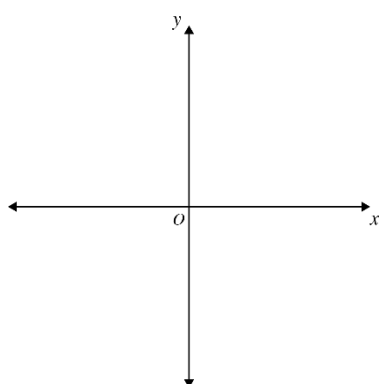
c $x > 3$



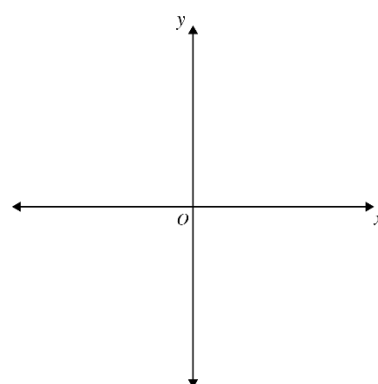
d $x \leq 0$



e $-1 \leq x \leq 1$



f All real x values



3 Find the range of these functions. Use a sketch if necessary.

a $f(x) = 2 - x$

b $f(x) = 5^x$

c $f(x) = 5^{-x}$

d $f(x) = \frac{2}{x}$

e $f(x) = 2 - x^2$

f $f(x) = x^2 + 4x$

NATURAL DOMAIN

- A function's **natural domain** includes all possible x -values that produce valid y -values.
When finding the natural domain, consider that
 - Zero denominators are undefined.**
 - Numbers in square roots must be non-negative.**

Example Identify natural domain from the equation of a function.

Find the natural domain of: $y = \frac{1}{x-2}$ $x-2 \neq 0$ $x \neq 2$ Domain: all real numbers except $x = 2$ $(-\infty, 2) \cup (2, \infty)$	Why is $x - 2 \neq 0$ for this function? <i>If $x - 2 = 0$, then you would be by zero, which is not possible.</i> Note: another way to write $x \in \mathbb{R} \mid x \neq 2$ is $(-\infty, 2) \cup (2, \infty)$
$y = \sqrt{x-1}$ $x-1 \geq 0$ $x \geq 1$ Domain: $x \geq 1$ $[1, \infty)$	Why is $x - 1 \geq 0$ for this function? <i>If $x - 1 < 0$, then you would be square rooting a number, which is not possible.</i>

Find the natural domain of each function.

a $y = x^2$ all real x $x \in \mathbb{R}$	b $f(x) = \sqrt{x-5}$ $x \geq 5$ $[5, \infty)$
c $f(x) = \frac{1}{3x}$ all real x except $x = 0$ $(-\infty, 0) \cup (0, \infty)$	d $y = \frac{1}{\sqrt{3-x}}$ $x < 3$ $(-\infty, 3)$

Key ideas

- To find the natural domain of a function, consider that:
 - You cannot divide by
 - You cannot square root a number.

1 Find the natural domain of these functions.

a $f(x) = 4x$

i $f(x) = 3x$

q $f(x) = x$

b $f(x) = \frac{1}{4x}$

j $f(x) = \sqrt{3x}$

r $f(x) = \frac{1}{\sqrt{x}}$

c $f(x) = \frac{1}{x+4}$

k $f(x) = \sqrt{x+3}$

s $f(x) = \frac{1}{\sqrt{5x}}$

d $f(x) = \frac{1}{-x+4}$

l $f(x) = \sqrt{-x+3}$

t $f(x) = \frac{1}{\sqrt{5x+1}}$

e $f(x) = \frac{1}{x-4}$

m $f(x) = \sqrt{x-3}$

u $f(x) = \frac{2}{\sqrt{5x+1}}$

f $f(x) = \frac{1}{6x-4}$

n $f(x) = \sqrt{8x-3}$

v $f(x) = \frac{2}{\sqrt{5x-1}}$

g $f(x) = \frac{3}{6x-4}$

o $f(x) = 5\sqrt{8x-3}$

w $f(x) = \frac{2}{\sqrt{1-5x}}$

h $f(x) = -\frac{3}{6x-4}$

p $f(x) = -5\sqrt{8x-3}$

x $f(x) = -\frac{2}{\sqrt{1-5x}}$

- 2** We can find the domain of functions with x 's in the denominator by letting the denominator equal and then excluding these values.
Find the domain of these functions.

a $f(x) = \frac{1}{2x-3}$

b $f(x) = \frac{2}{x+7} + 3$

c $f(x) = \frac{1}{(x-3)^2}$

- 3** We can find the domain of functions with x 's inside a square root by setting up an inequality with the inside greater or equal to
Find the domain of these functions

a $f(x) = \sqrt{x-2}$

b $f(x) = \sqrt{5x-4}$

c $f(x) = \sqrt{7-2x}$

- 4** Identify the natural domain of these functions.

a $f(x) = \frac{1}{x}$

b $f(x) = \frac{1}{x-2}$

c $f(x) = \frac{1}{x+3}$

d $f(x) = \frac{3}{x-1}$

e $f(x) = \frac{3}{2x+1}$

f $f(x) = \frac{-1}{1-x}$

g $f(x) = \sqrt{x}$

h $f(x) = \sqrt{x-2}$

i $f(x) = \sqrt{x+2}$

j $f(x) = \sqrt{2-x}$

k $f(x) = \sqrt{2x+1}$

l $f(x) = \sqrt{2-3x}$

m $f(x) = \frac{2}{\sqrt{3x-9}}$

n $f(x) = \frac{3}{\sqrt{4-2x}}$

o $f(x) = \frac{x+1}{\sqrt{5x-13}}$

5

- a** Find the solutions to the equation $(x - 1)(x - 2) = 0$
- b** Hence, find the natural domain of $f(x) = \frac{1}{(x-1)(x-2)}$
- c** Use a similar technique to find the domain of the following functions.

$$f(x) = \frac{2}{(x+1)(x-3)}$$

$$f(x) = \frac{3}{(2x-1)(3x-1)}$$

$$f(x) = \frac{x}{x^2-4x+3}$$

- 6** Find the natural domain of $f(x) = \frac{1}{\sqrt{4x^2-9}}$

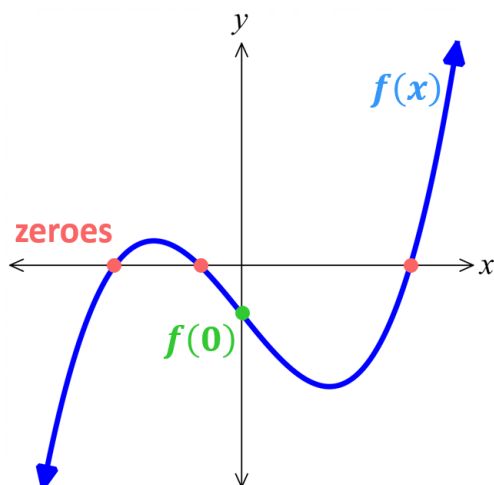
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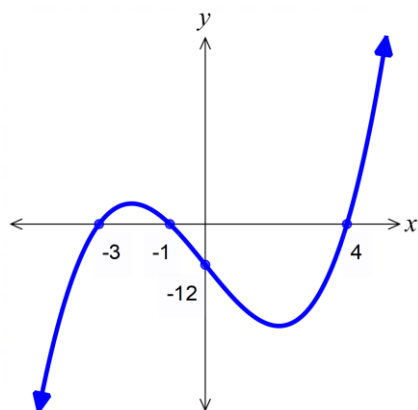
LANGUAGE OF FUNCTIONS

- A **zero** of a function is the x value where $f(x) = 0$. (i.e., the x -intercepts).
- The y -intercept is the value of $f(0)$.
- A function can have many zeroes but only one $f(0)$.



1 Identify the zeroes of these functions, and the value of $f(0)$.

a



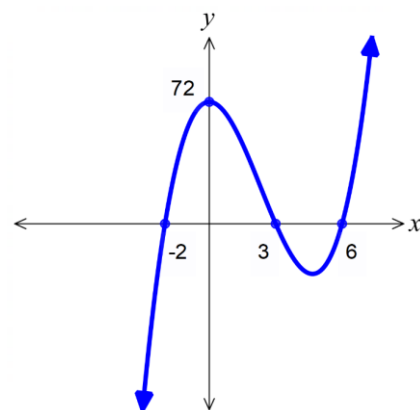
Zeroes:

$f(0) =$

$$x = -3, -1, 4$$

$$f(0) = -12$$

b



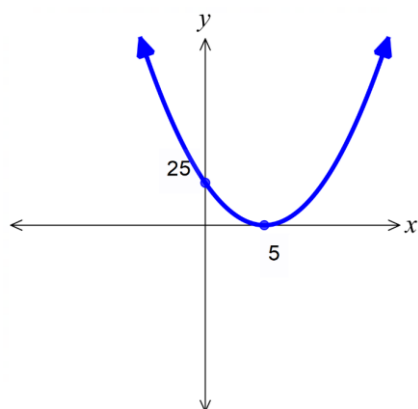
Zeroes:

$f(0) =$

$$x = -2, 3, 6$$

$$f(0) = 72$$

c



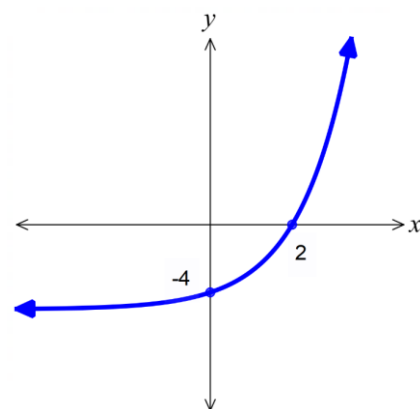
Zeroes:

$f(0) =$

$$x = 5$$

$$f(0) = 25$$

d



Zeroes:

$f(0) =$

$$x = 2$$

$$f(0) = -4$$

TRANSLATIONS OF FUNCTIONS

Investigation Translations of functions.

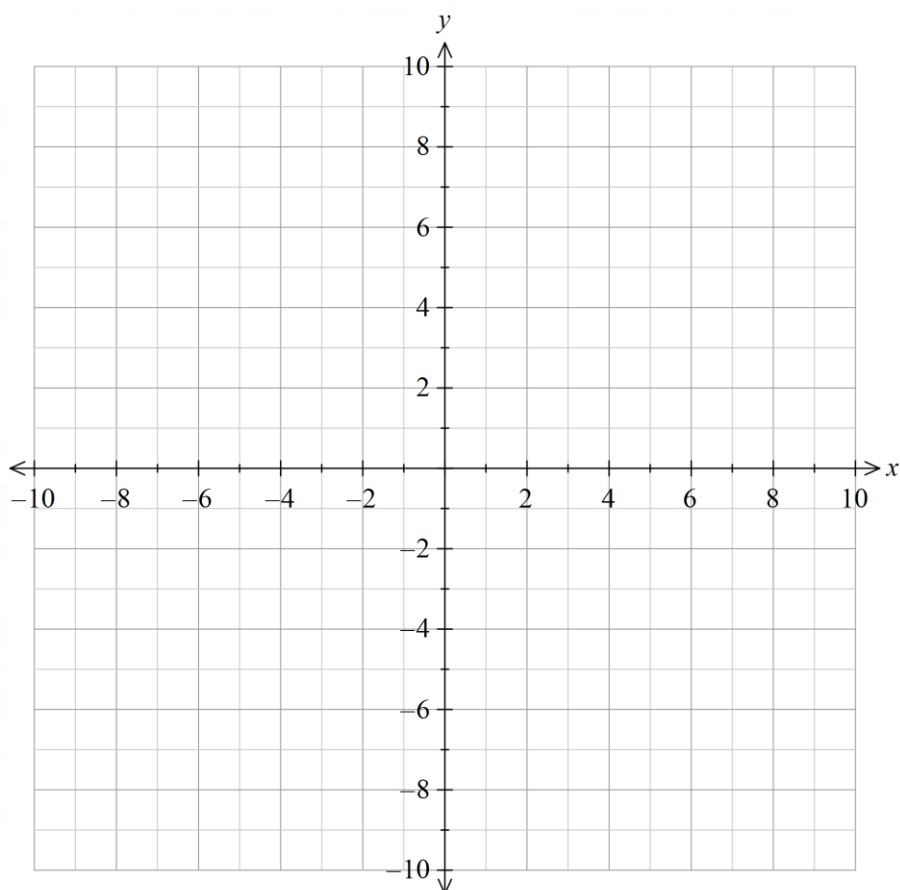
Consider the function $f(x) = x^2$.

Write the equation, then graph the function on the same axis as the original.

$y = f(x) + 1 =$

$y = f(x) - 3 =$

x	-3	-2	-1	0	1	2	3
$y = f(x)$							
$y = f(x) + 1$							
$y = f(x) - 3$							



$f(x) + c$ is the original function, $f(x)$, translated v..... by c

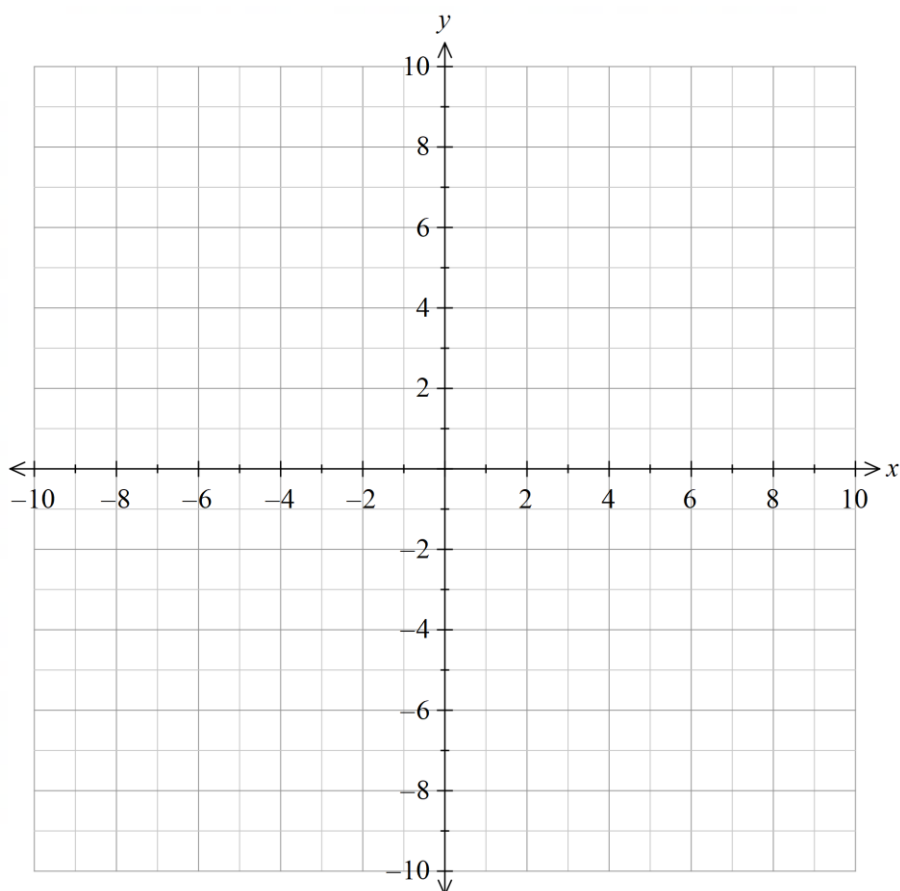
Consider the function $f(x) = x^2$.

Write the equation, then graph the function on the same axis as the original.

$y = f(x + 1) =$

$y = f(x - 2) =$

x	-3	-2	-1	0	1	2	3
$y = f(x)$							
$y = f(x + 1)$							
$y = f(x - 2)$							



$f(x - b)$ is the original function, $f(x)$, translated h..... by b

Observe that the values of $(x - 2)^2$ are the values of x^2 shifted 2 units to the **right**.

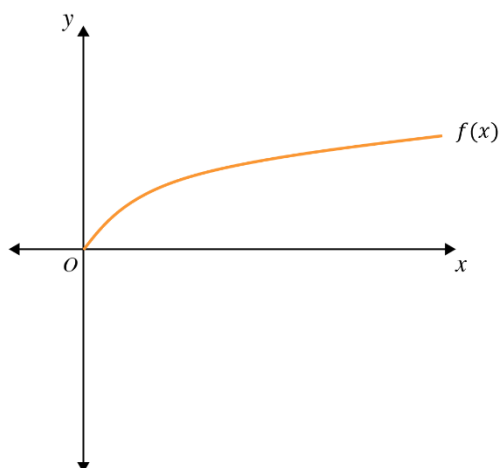
This is the opposite to what you would expect.

By **subtracting 2** from x , the new function reaches the same y -value **2 units later**.

TRANSLATION OF FUNCTIONS

$$y = f(x)$$

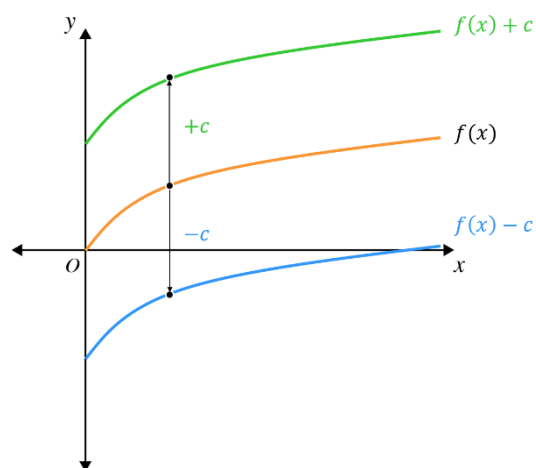
The original graph.



$$y = f(x) + c$$

All y values are shifted by c .

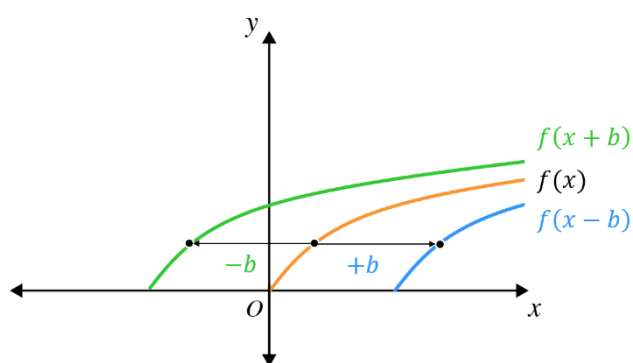
Replace original y values with $y + c$.



$$y = f(x - b)$$

All x values are shifted by b .

Replace original x values with $x - b$.



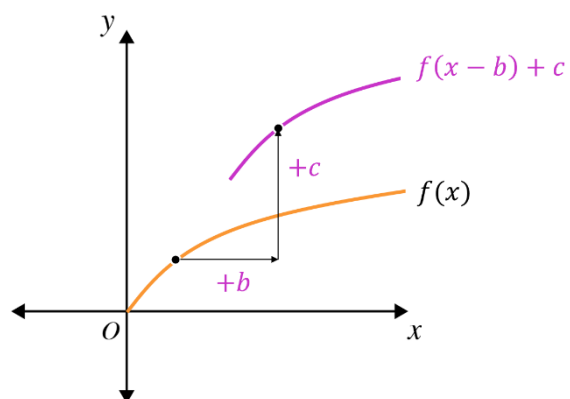
Vertical translation



$$y = f(x - b) + c$$

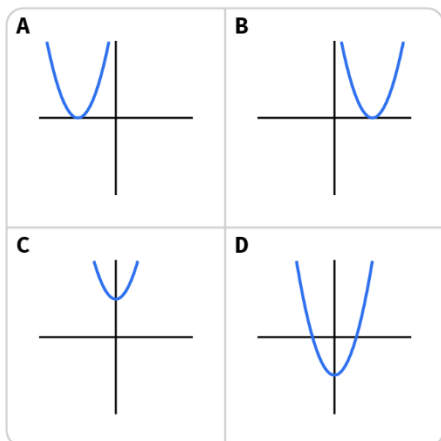


Horizontal translation

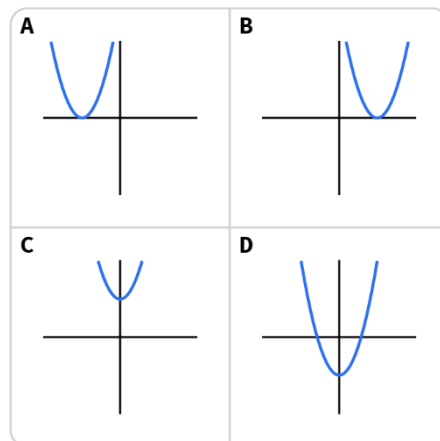


Check your understanding

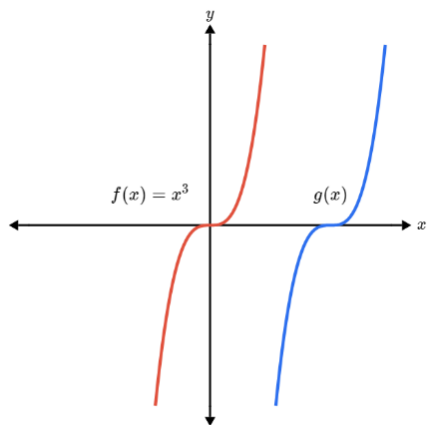
Consider the parabola $f(x) = x^2$
Which of these is the graph of $y = f(x + 3)$?



Consider the parabola $f(x) = x^2$
Which of these is the graph of $y = f(x) - 3$?



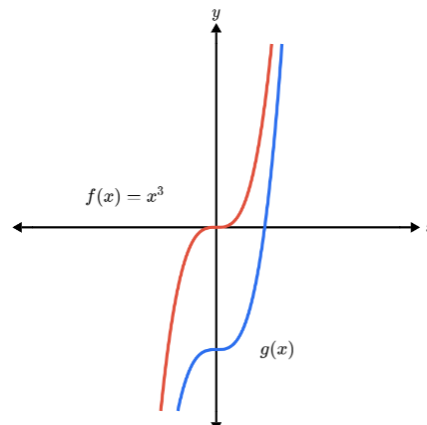
Here are graphs of $f(x) = x^3$ and $g(x)$.



What is the equation for $g(x)$?

- A. $g(x) = x^3 + 4$
- B. $g(x) = x^3 - 4$
- C. $g(x) = (x + 4)^3$
- D. $g(x) = (x - 4)^3$

Here are graphs of $f(x) = x^3$ and $g(x)$.

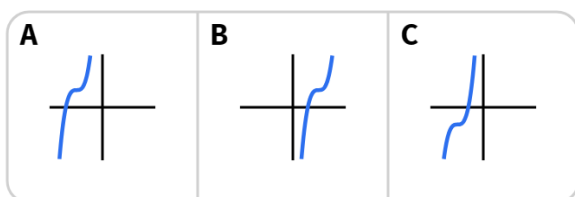


What is the equation for $g(x)$?

- A. $g(x) = x^3 + 4$
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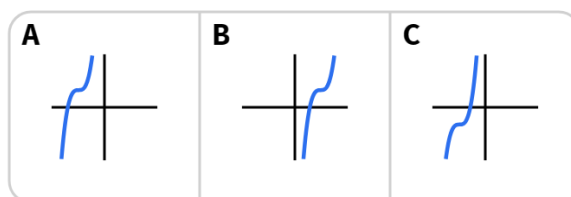
Consider the cubic curve $f(x) = x^3$
Which is the graph of $g(x)$, given

$$g(x) = (x + 4)^3 + 3$$

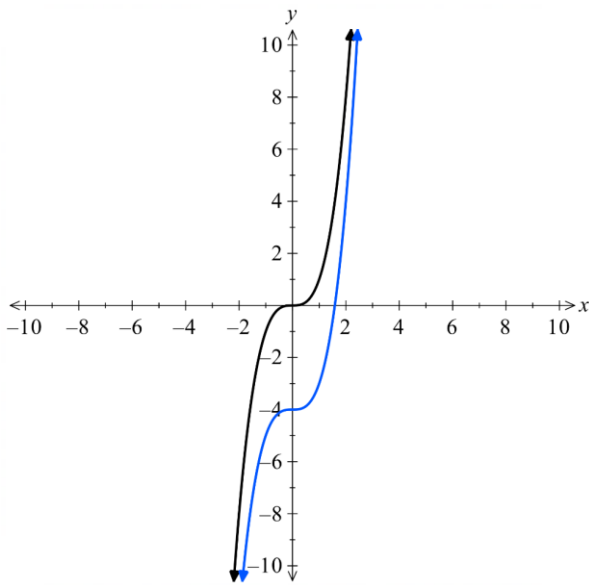
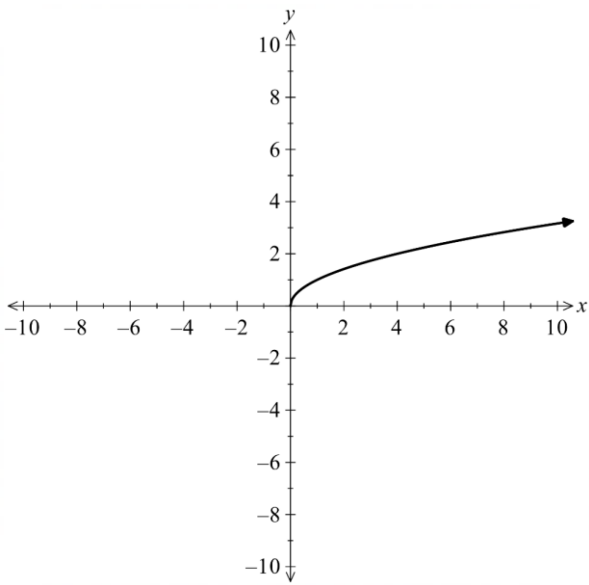
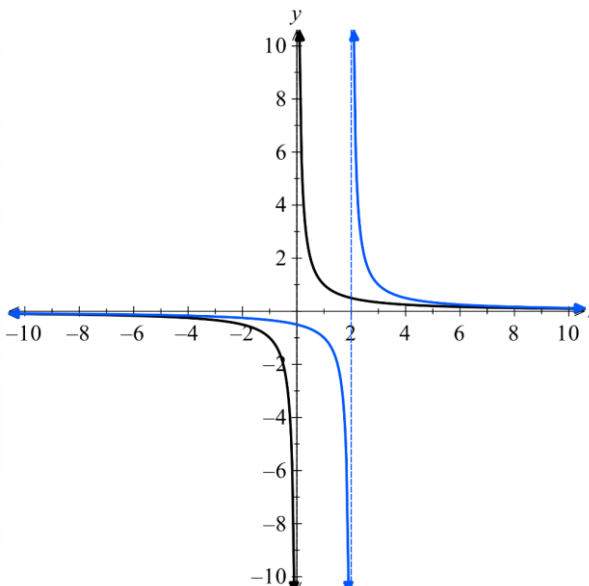
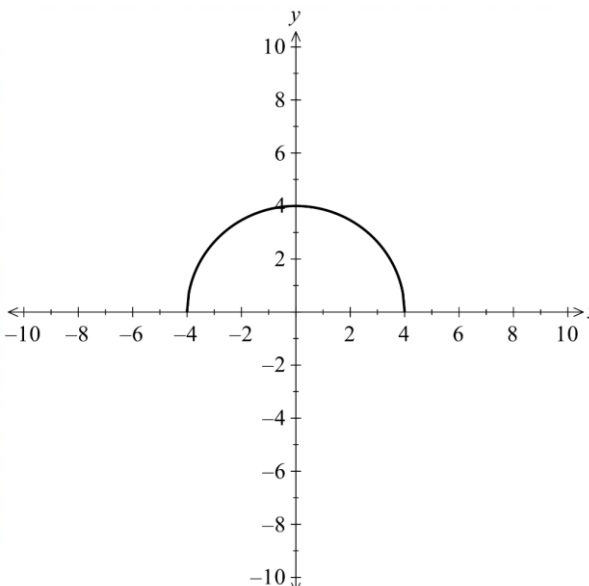


Consider the cubic curve $f(x) = x^3$
Which is the graph of $g(x)$, given

$$g(x) = (x - 3)^3 + 2$$



Example Sketch graphs using horizontal and vertical transformations.

Example	Your Turn
Given $y = f(x)$, sketch $y = f(x) - 4$ Curve shifted down by 4.  A Cartesian coordinate system with x and y axes ranging from -10 to 10. A black curve, representing $y = f(x)$, passes through the origin (0,0) and has a vertical asymptote at $x = 2$. A blue curve, representing $y = f(x) - 4$, is shifted 4 units down from the black curve. It passes through (0, -4) and has a vertical asymptote at $x = 2$. A dashed blue line indicates the asymptote at $x = 2$.	Given $f(x)$, sketch $y = f(x) + 3$  A Cartesian coordinate system with x and y axes ranging from -10 to 10. A black curve, representing $y = f(x)$, starts at the origin (0,0) and increases as x increases, passing through approximately (10, 3). A blue curve, representing $y = f(x) + 3$, is shifted 3 units up from the black curve. It starts at (0, 3) and increases as x increases, passing through approximately (10, 6).
Given $y = f(x)$, sketch $y = f(x - 2)$ Curve shifted right by 2.  A Cartesian coordinate system with x and y axes ranging from -10 to 10. A black curve, representing $y = f(x)$, has a vertical asymptote at $x = 0$ and passes through the origin (0,0). A blue curve, representing $y = f(x - 2)$, is shifted 2 units right from the black curve. It has a vertical asymptote at $x = 2$ and passes through (2, 0). A dashed blue line indicates the asymptote at $x = 2$.	Given $y = f(x)$, sketch $y = f(x + 3)$  A Cartesian coordinate system with x and y axes ranging from -10 to 10. A black curve, representing $y = f(x)$, is a semi-circle centered at (0, 4) with a radius of 4, defined on the interval $x \in [-4, 4]$. A blue curve, representing $y = f(x + 3)$, is shifted 3 units left from the black curve. It is a semi-circle centered at (-3, 4) with a radius of 4, defined on the interval $x \in [-7, 1]$.

Key ideas

- Let $f(x)$ be a function and $b, c > 0$,
 - $y = f(x) + c$ is translated c units up | down from the original function.
 - $y = f(x) - c$ is translated c units up | down from the original function.
 - $y = f(x - b)$ is translated b units left | right from the original function.
 - $y = f(x + b)$ is translated b units left | right from the original function.

1 Given $y = x^2$, write the translation that is required to obtain the graphs of the following:

a $y = x^2 + 2$

b $y = x^2 - 3$

c $y = (x + 1)^2$

d $y = (x - 2)^2$

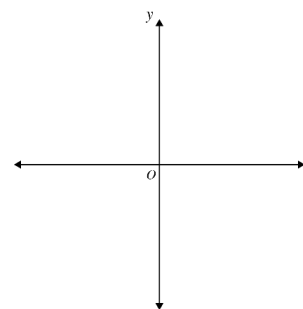
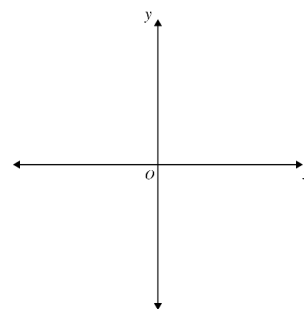
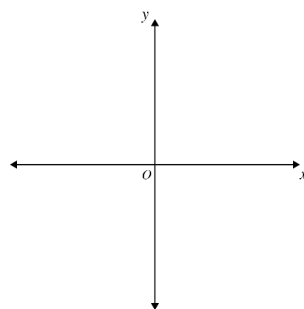
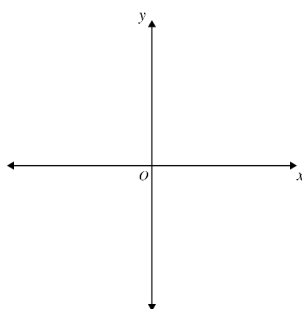
2 Let $f(x) = x^2$. Sketch these graphs, labelling the intercepts.

a $y = f(x) + 1$

b $y = f(x + 1)$

c $y = f(x) - 4$

d $y = f(x - 2)$



3 Given $y = 2^x$, write the translation that is required to obtain the graphs of the following:

a $y = 2^x - 1$

b $y = 2^{x-2}$

c $y = 2^x + 1$

d $y = 2^{x+1}$

4 Choose the word *left*, *right*, *up* or *down* to complete each sentence.

a The graph of $y = 2^x + 1$ is the translation of the graph of $y = 2^x$ by 1 unit.

b The graph of $y = 2^x - 3$ is the translation of the graph of $y = 2^x$ by 3 units.

c The graph of $y = 2^{x-4}$ is the translation of the graph of $y = 2^x$ by units.

d The graph of $y = 2^{x+1}$ is the translation of the graph of $y = 2^x$ by unit.

e The graph of $y = x^2 + 2$ is the translation of the graph of $y = x^2$ by unit.

f The graph of $y = x^2 - 5$ is the translation of the graph of $y = x^2$ by unit.

g The graph of $y = (x + 4)^2$ is the translation of the graph of $y = x^2$ by unit.

h The graph of $y = (x - 3)^2$ is the translation of the graph of $y = x^2$ by unit.

i The graph of $y = \frac{1}{x-3}$ is the translation of the graph of $y = \frac{1}{x}$ by units.

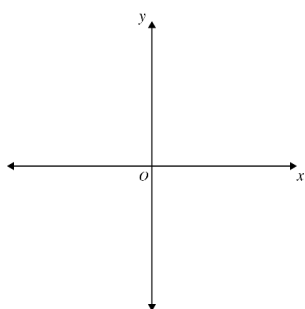
j The graph of $y = \frac{1}{x+2}$ is the translation of the graph of $y = \frac{1}{x}$ by units.

k The graph of $y = \frac{1}{x} + 3$ is the translation of the graph of $y = \frac{1}{x}$ by units.

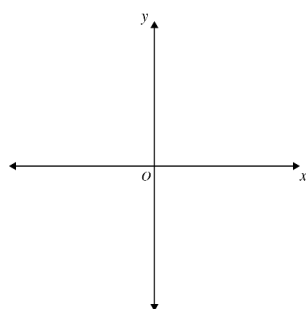
l The graph of $y = \frac{1}{x} - 6$ is the translation of the graph of $y = \frac{1}{x}$ by units.

5 Sketch each parabola by shifting $y = x^2$ either horizontally or vertically. Show the vertex.

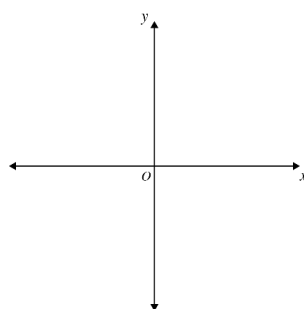
a $y = x^2 + 1$



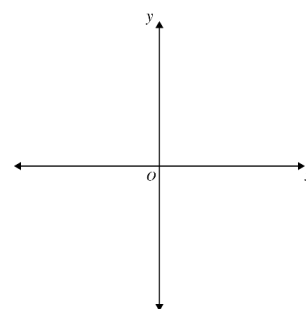
b $y = x^2 - 1$



c $y = (x - 1)^2$

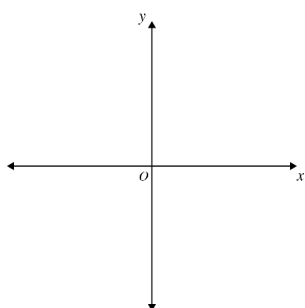


d $y = (x + 1)^2$

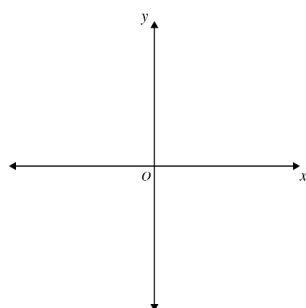


6 Sketch each hyperbola by shifting $y = \frac{1}{x}$ either horizontally or vertically. Show the asymptotes.

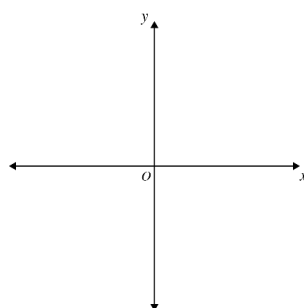
a $y = \frac{1}{x+1}$



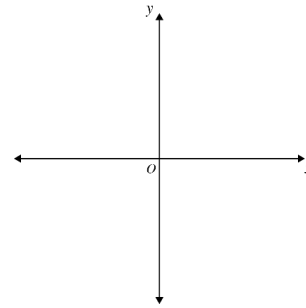
b $y = \frac{1}{x-1}$



c $y = \frac{1}{x} + 1$



d $y = \frac{1}{x} - 1$



7 Write down the new equation obtained when the following translation is applied to $f(x)$

a $f(x) = 2x$
left 2 units

b $f(x) = 2x$
down 3 units

c $f(x) = \frac{1}{x-1}$
left 2 units

d $f(x) = \frac{1}{x-1}$
down 3 units

e $f(x) = 2x^2$
right 4 units

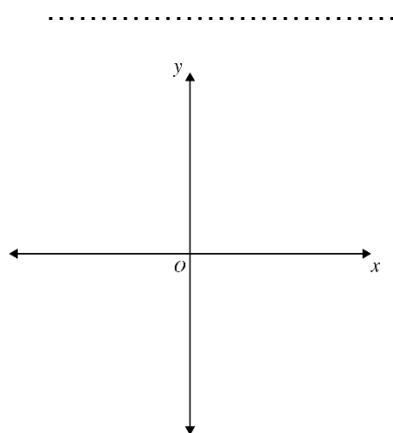
f $f(x) = 2x^2$
up 2 units

g $f(x) = 3 - 2x$
up 2 units

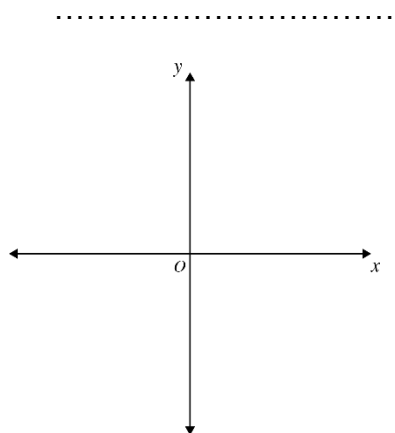
h $f(x) = 3 - 2x$
left 3 units

- 8** Write down the new equation for each function or relation after the given translation has been applied. Then sketch the graph of the new curve.

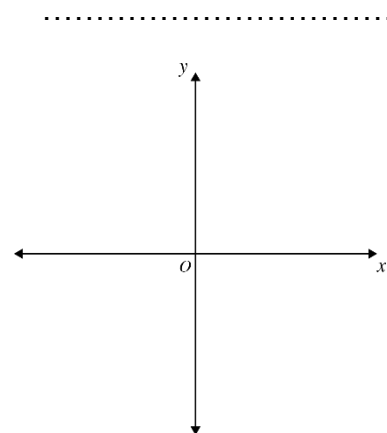
a $y = x^2$ right 1 unit



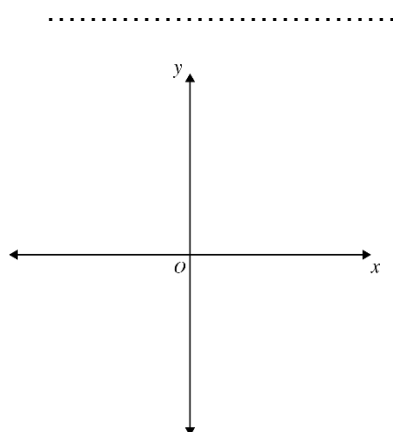
b $y = 2^x$ down 3 units



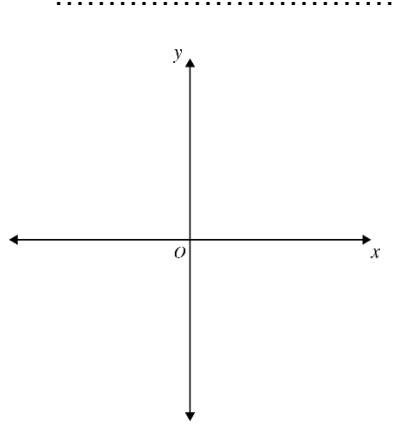
c $y = x^3$ left 1 unit



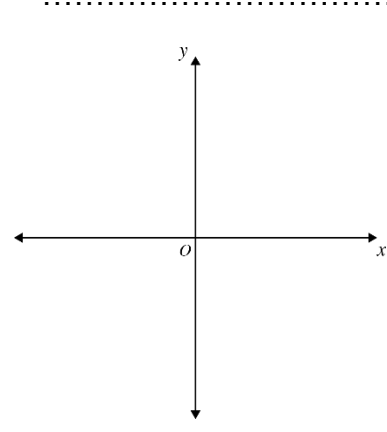
d $y = \frac{1}{x}$ right 3 units



e $y = x^2 - 4$ left 1 unit

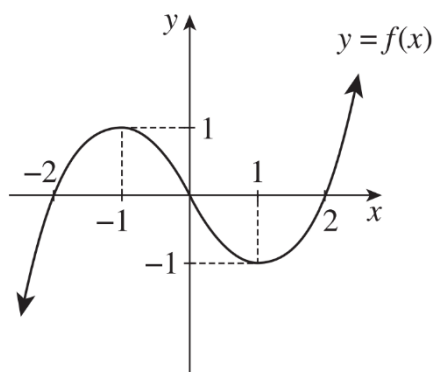


f $xy = 1$ down 1 unit

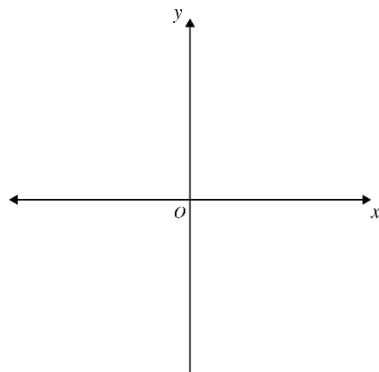


- 9** In each case an unknown function has been drawn. Draw the functions specified, labelling all key points.

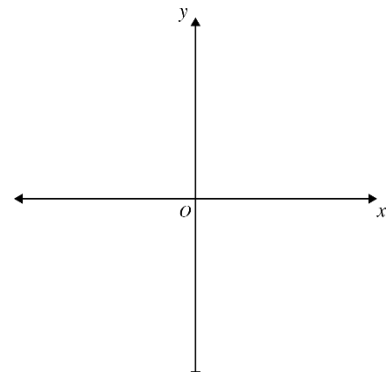
a $y = f(x)$



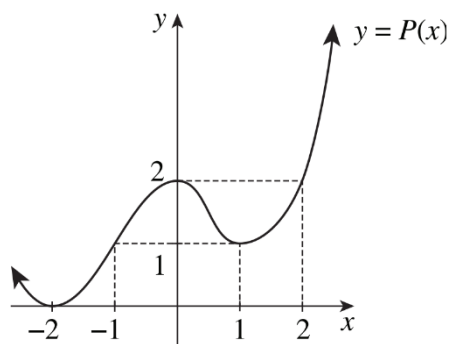
$y = f(x - 2)$



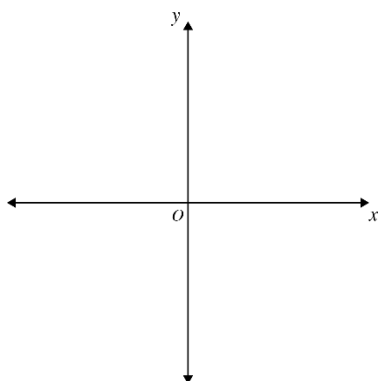
$y = f(x + 1)$



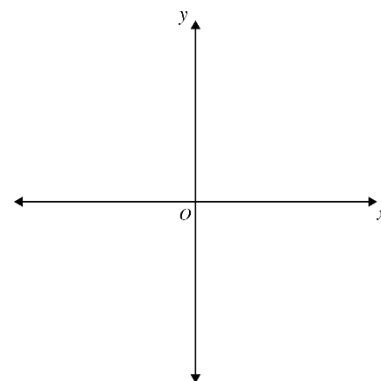
b $y = P(x)$



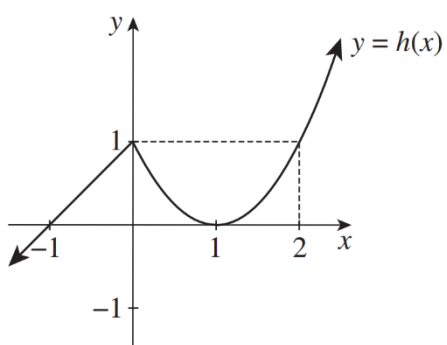
$y = P(x + 2)$



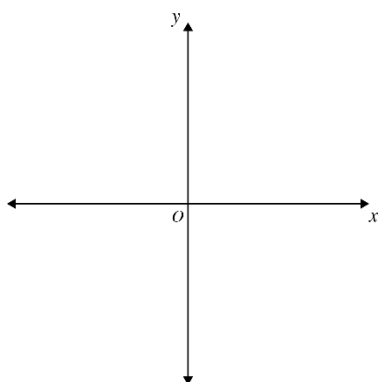
$y = P(x + 1)$



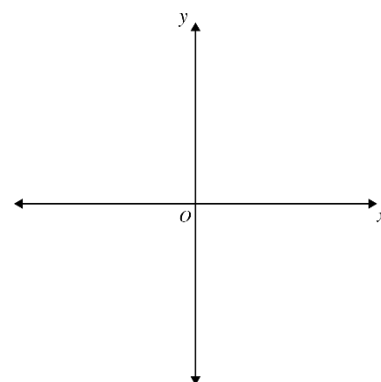
c $y = h(x)$



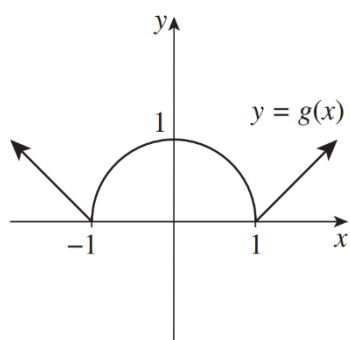
$y = h(x) + 1$



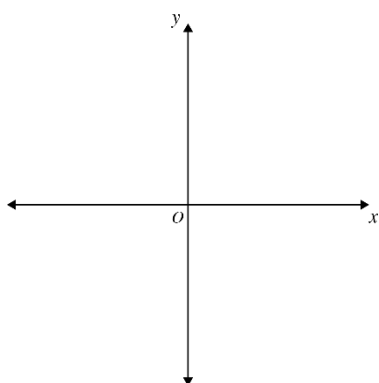
$y + 1 = h(x)$



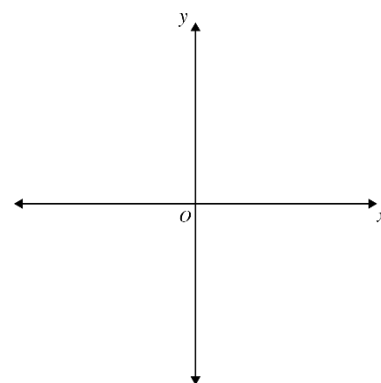
d $y = g(x)$



$y = g(x - 1)$



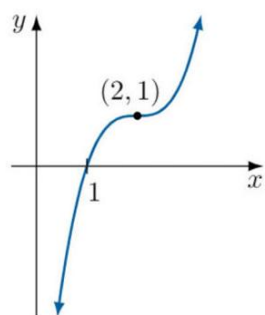
$y - 1 = g(x)$



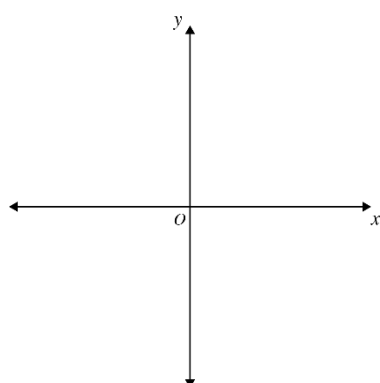
10 Each sketch shows the graph of the equation.

For each part, describe the transformation, write the new equation, and sketch the curve.

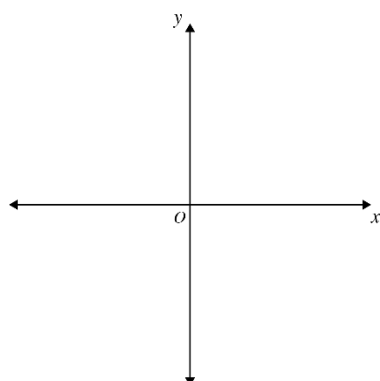
a $y = (x - 2)^3 + 1$



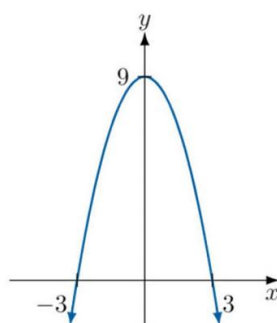
i $y = f(x + 1)$



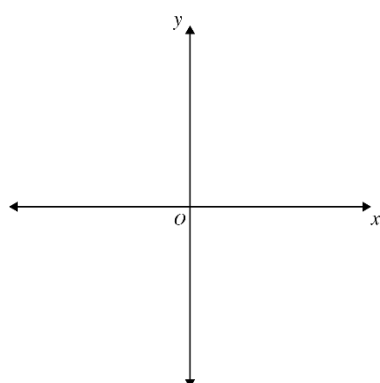
ii $y = f(x) + 1$



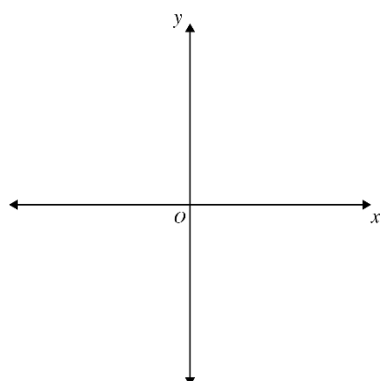
b $f(x) = (3 - x)(3 + x)$



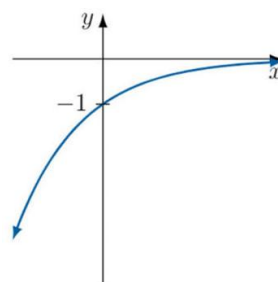
i $y = f(x - 1)$



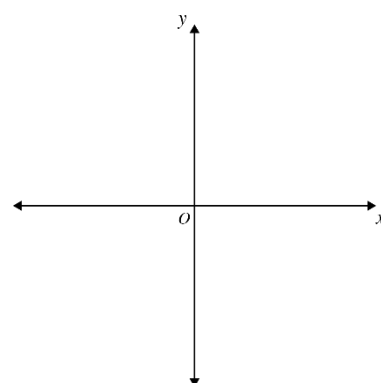
ii $y = f(x) - 1$



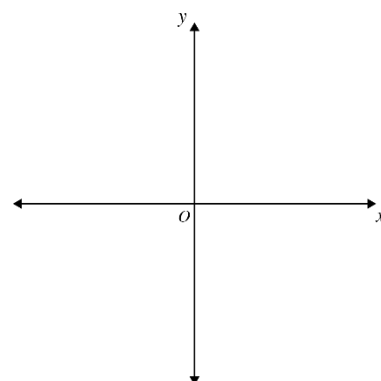
c $f(x) = -2^{-x}$



i $y = f(x + 2)$

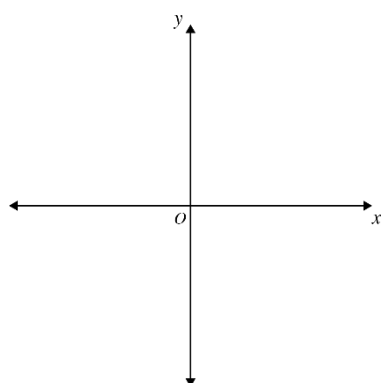


ii $y = f(x) + 2$

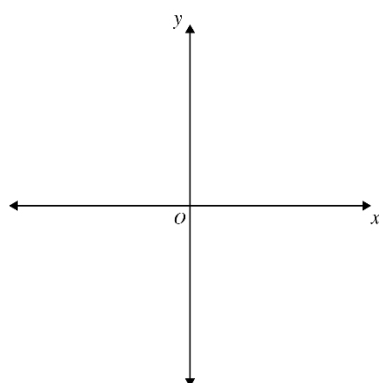


- 11** In each part, complete the square in the quadratic. Then identify its graph as a translation of a parabola with vertex at the origin. Finally, sketch its graph.

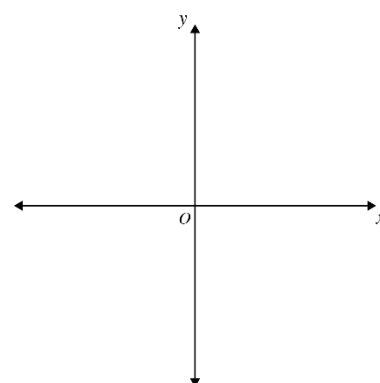
a $y = x^2 + 2x + 3$



b $y = -x^2 + 4x + 1$

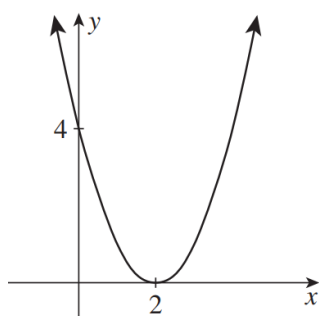


c $y = 2x^2 - 4x - 2$

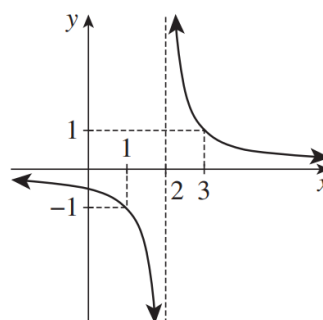


- 12** Describe each graph below as the parabola $y = x^2$ or the hyperbola $xy = 1$ transformed by shifts, and hence write down its equation.

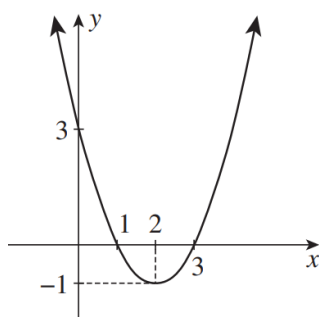
a



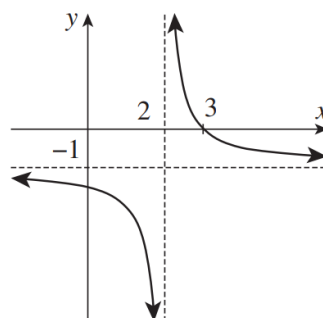
b



c



d

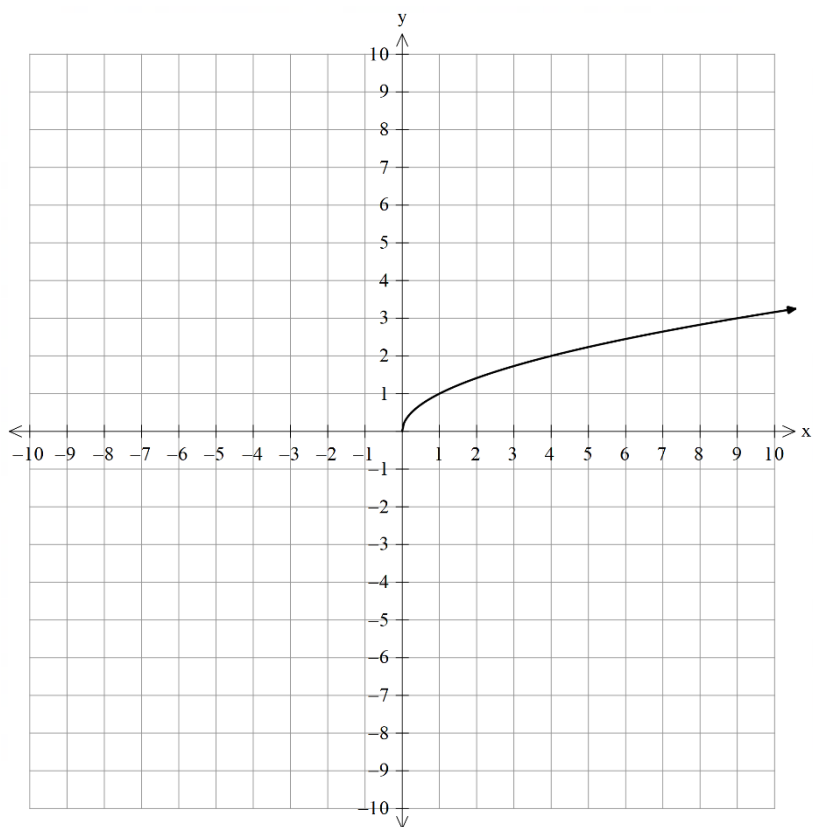


13 The graph shows $y = \sqrt{x}$, use translations to sketch on the same axis:

a $y = \sqrt{x+4}$

b $y = \sqrt{x} + 4$

c $y = \sqrt{x+4} - 2$

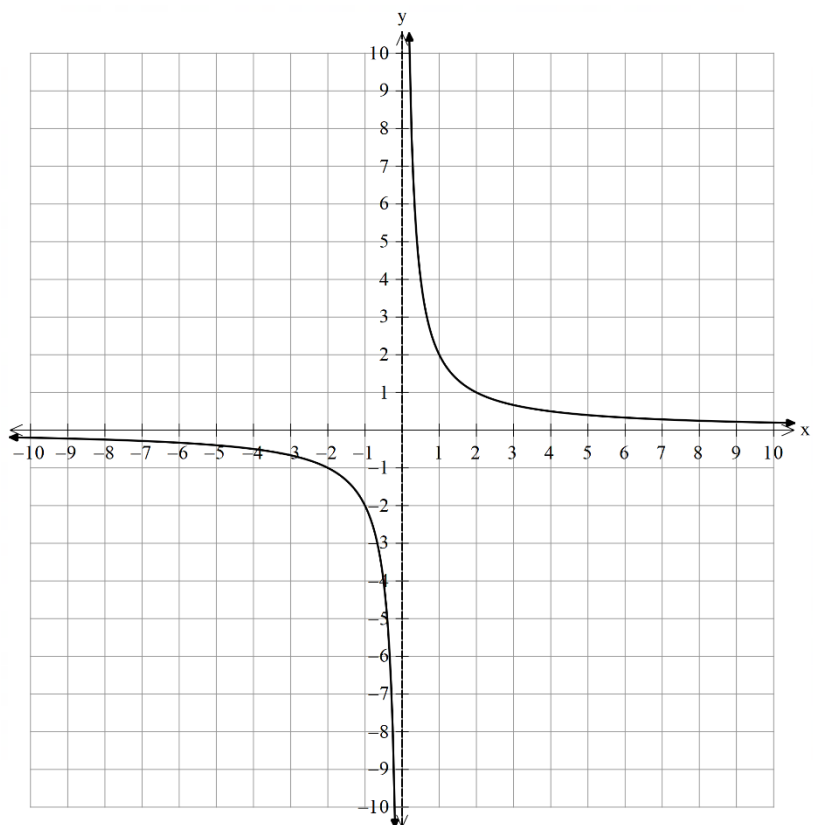


14 The graph shows $y = \frac{2}{x}$, use translations to sketch on the same axis:

a $y = \frac{2}{x} + 1$

b $y = \frac{2}{x+2}$

c $y = \frac{2}{x+2} + 1$



15 Be careful with negative signs.

Write the new equations if each of these curves are translated to the right by 2 units.

a $f(x) = 3 - 2x$

b $f(x) = 3 - 4x - x^2$

c $f(x) = 2^{-x}$

16 If the coefficient of x is not 1, translations are not so straightforward.

This idea will be developed further in Year 12.

Consider the graph of $f(x) = (2x + 1)^2$ and $g(x) = (2x - 5)^2$

Joan claims that $y = g(x)$ is the graph of $y = f(x)$ translated right by 6 units.

a Show that Joan is incorrect by translating $f(x)$ to the right 6 units by replacing x with $x - 6$

b Write the equation of the graph of $y = f(x)$ if it is shifted to the right by a units.

c Hence determine how many units to the right $f(x)$ needs to be translated to obtain the graph of $g(x)$

d For each of the following, in what direction was $f(x)$ translated to yield $g(x)$ and by how many units?

i $y = \sqrt{2x - 3}, y = \sqrt{2x + 7}$

ii $y = (3x + 5)^2, y = (3x - 1)^2$

iii $y = 5^{-x}, y = 5^{-x+1}$

iv $y = e^{-2x}, y = e^{-2x+4}$

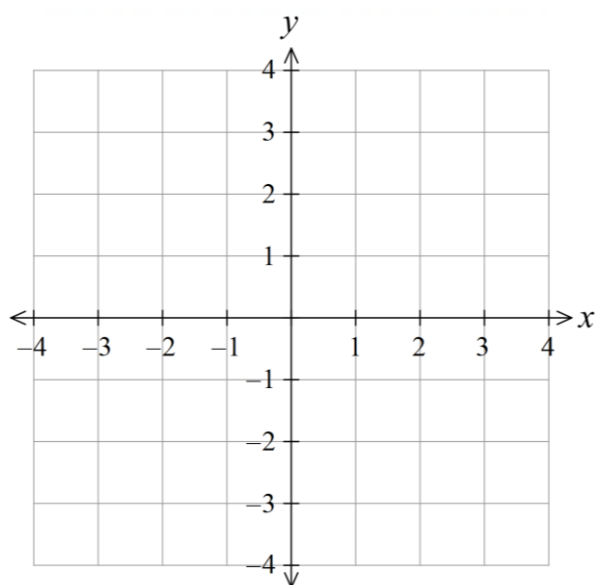
REFLECTIONS OF FUNCTIONS

Investigation Reflections of Functions

Consider the function $f(x) = x^2 + 2x$. Write the equation and graph using a table of values:

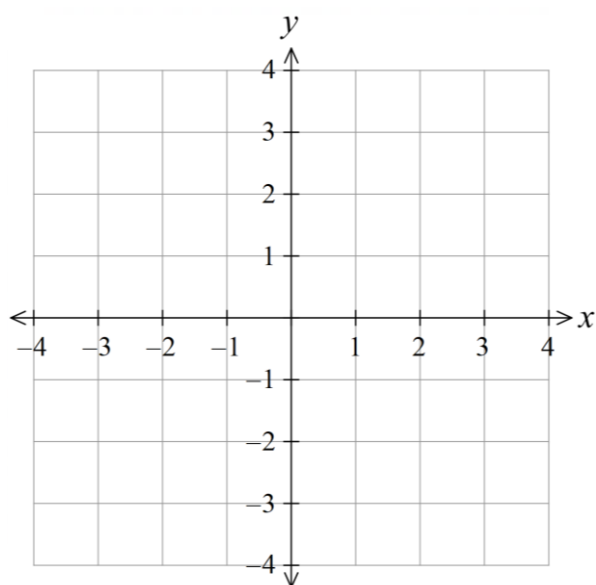
$y = f(x) = \dots\dots\dots$

x	-3	-2	-1	0	1	2	3
y							



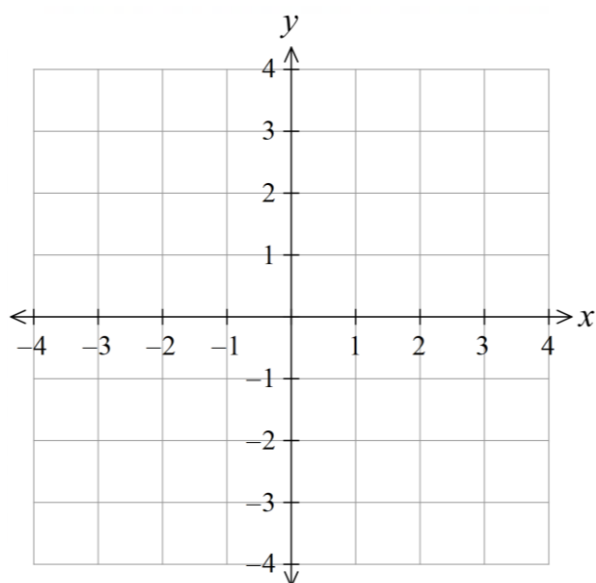
$y = -f(x) = \dots\dots\dots$

x	-3	-2	-1	0	1	2	3
y							



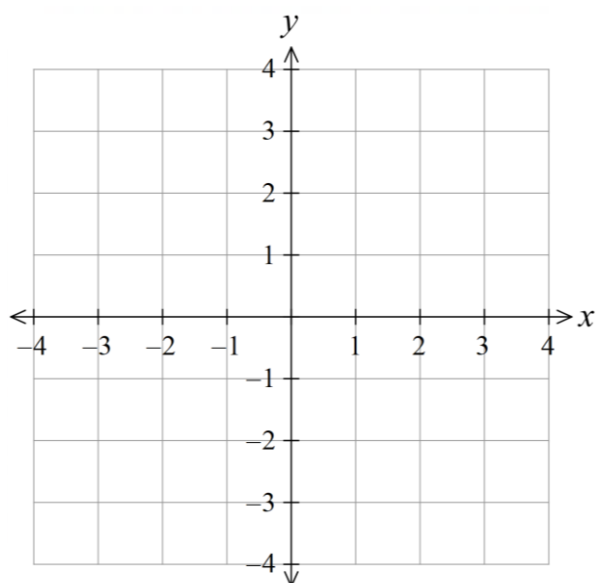
$y = f(-x) = \dots\dots\dots$

x	-3	-2	-1	0	1	2	3
y							

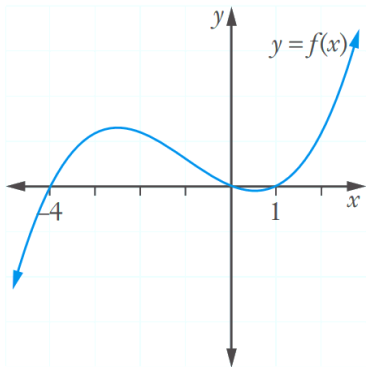
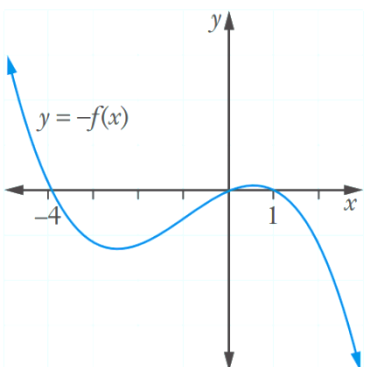
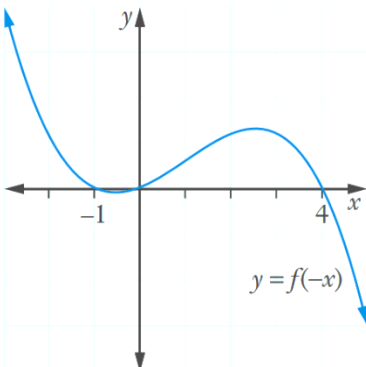
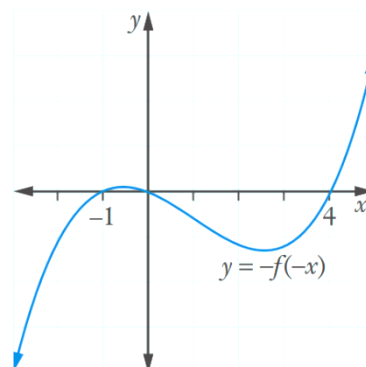


$y = -f(-x) = \dots\dots\dots$

x	-3	-2	-1	0	1	2	3
y							



REFLECTIONS OF FUNCTIONS

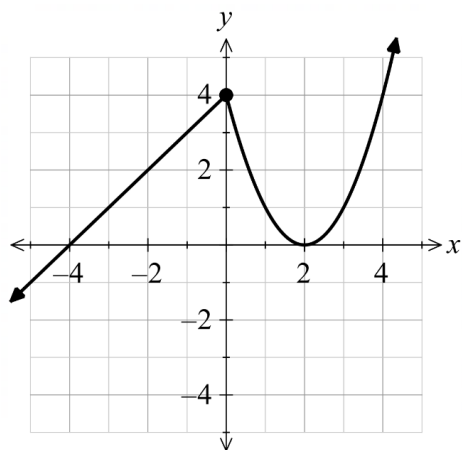
$y = f(x)$	$y = -f(x)$
The original graph 	All y values are replaced by $-y$ Reflect the graph across the x axis. 
$y = f(-x)$	$y = -f(-x)$
All x values are replaced with $-x$ Reflect the graph across the y axis. 	All x values are replaced with $-x$ All y values are replaced by $-y$ Reflect the graph across both axes. (rotate 180°) 

- If you cannot visualise the graph after reflection, follow these steps:
 - Determine the new location of x and y intercepts.
 - If specific coordinates are given, determine their new location.
 Use these rules:

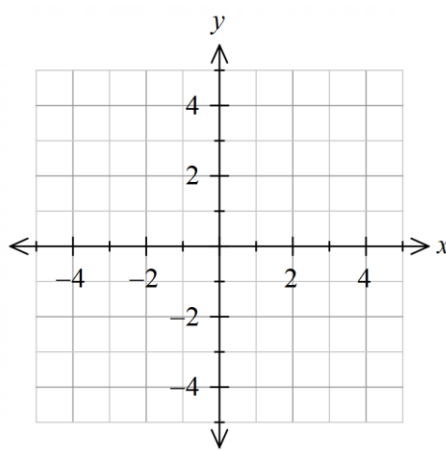
Reflection across x -axis $-f(x)$	Reflection across y -axis $f(-x)$	Reflection across both (rotation by 180°) $-f(-x)$
$A(x, y)$ ↓ $A'(x, -y)$	$A(x, y)$ ↓ $A'(-x, y)$	$A(x, y)$ ↓ $A'(-x, -y)$

- a** The graph of $y = f(x)$ is given. Sketch $y = -f(x)$, $y = f(-x)$ and $y = -f(-x)$

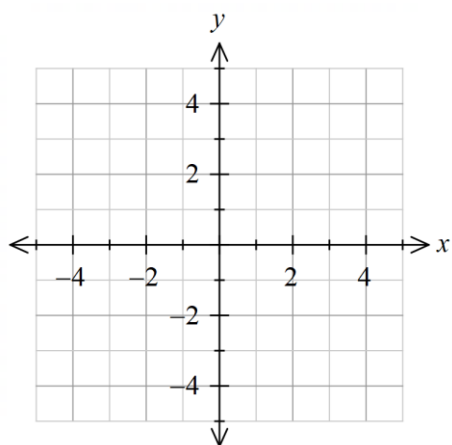
$$y = f(x)$$



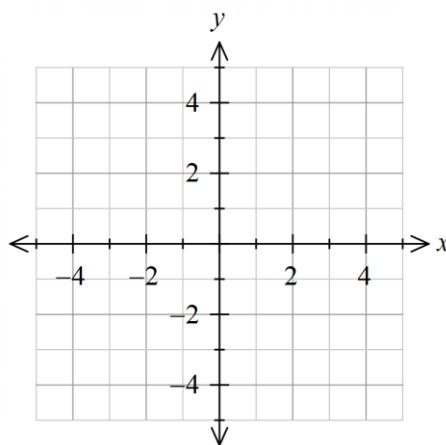
$$y = -f(x)$$



$$y = f(-x)$$

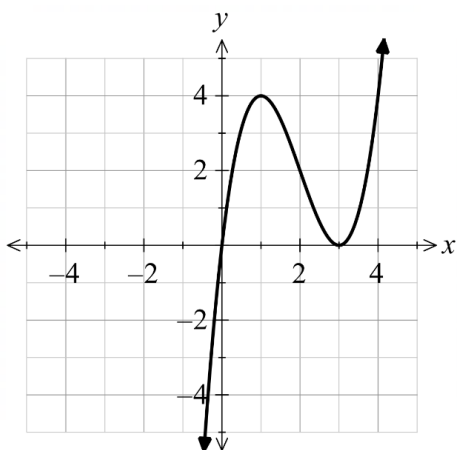


$$y = -f(-x)$$

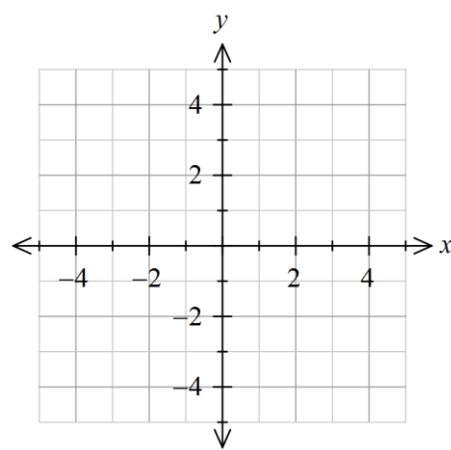


- b** The graph of $y = P(x)$ is given. Sketch $y = -P(x)$, $y = P(-x)$ and $y = -P(-x)$

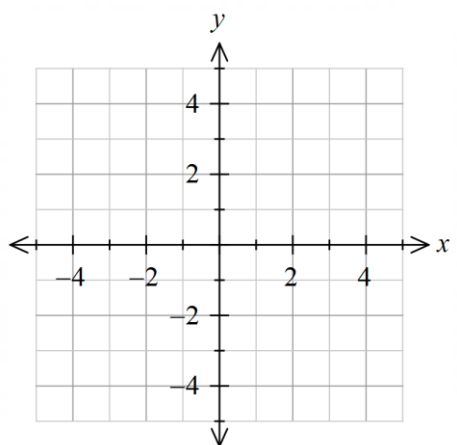
$$y = P(x)$$



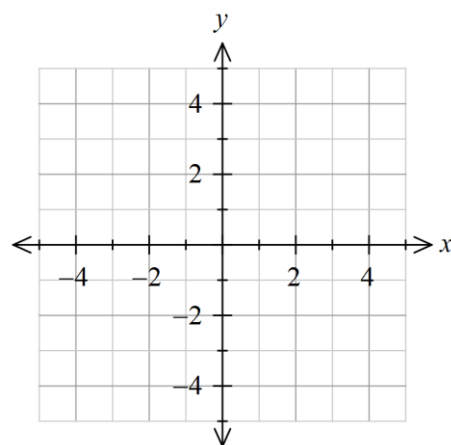
$$y = -P(x)$$



$$y = P(-x)$$



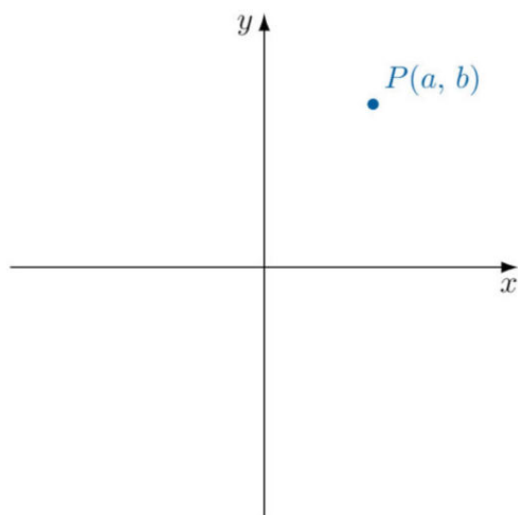
$$y = -P(-x)$$



Key ideas

- When $y = f(x)$ is reflected across the axis, we can write this as $y = f(-x)$.
 - Replace all x with
 - All positive x coordinates become
 - All negative x coordinates become
 - The coordinates stay the same.
- When $y = f(x)$ is reflected across the y axis, we can write this as $y = \dots\dots\dots$.
 - Replace y with
 - All positive y coordinates become
 - All negative y coordinates become
 - The coordinates stay the same.
- When $y = f(x)$ is reflected across both axes, we can write this as $y = \dots\dots\dots$, and this is the same as the graph by 180° .
 - Replace all x with
 - Replace all y with

- 1 The point $P(a, b)$ is shown in the diagram below.



- a The point P is reflected across the x -axis. Label the result Q and write the coordinates.

$$Q(a, -b)$$

- b The point P is reflected across the y -axis. Label the result R and write the coordinates.

$$R(-a, b)$$

- c The point R is reflected across the x -axis. Label the result S. and write the coordinates.

$$R(-a, -b)$$

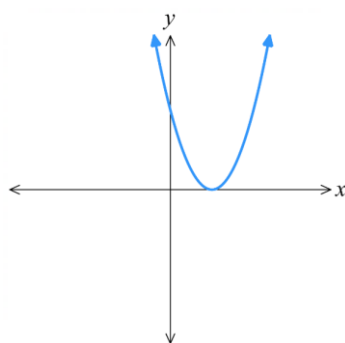
- d The point Q is reflected across the y -axis. What do you notice?

$$R(-a, -b)$$

Reflecting across x then y is the same as reflecting across y then x .

- 2 $f(x) = (x - 2)^2$

- a Sketch $y = f(x)$



- b Find the equation of:

$$y = -f(x)$$

$$y = f(-x)$$

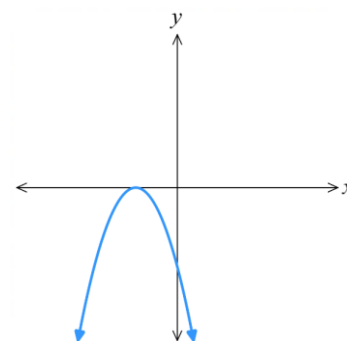
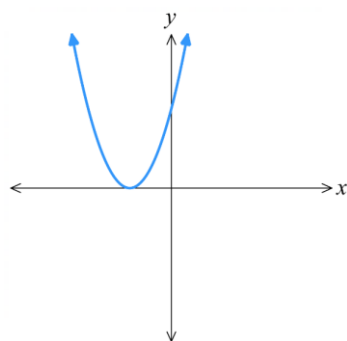
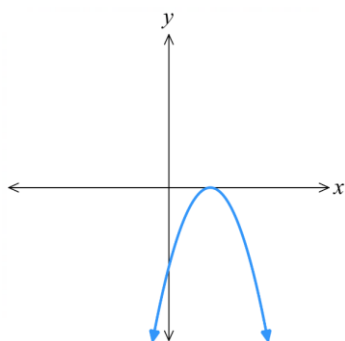
$$y = -f(-x)$$

- c hence sketch the graph.

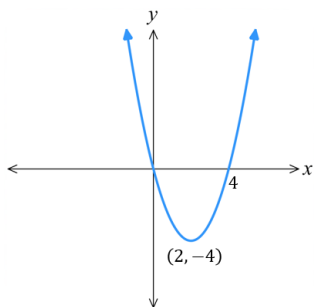
$$y = -(x - 2)^2$$

$$y = (-x - 2)^2 = (x + 2)^2$$

$$y = -(x + 2)^2$$



- 3 For the graph of $f(x) = x(x - 4)$,
 a Sketch the graph,
 showing intercepts and vertex.



- b find the equation of the graph after:

reflection across x axis

reflection across y axis

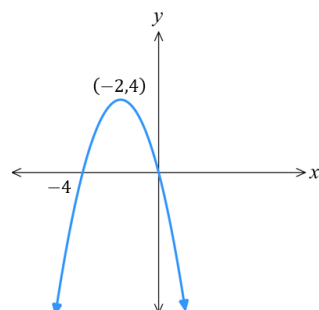
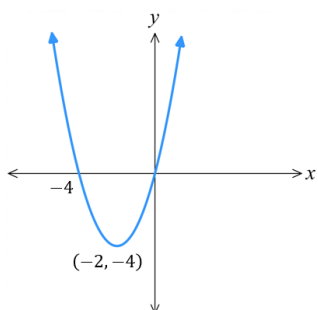
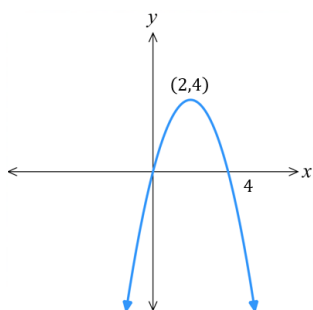
reflection across both axes

$$-f(x) = -x(x - 4)$$

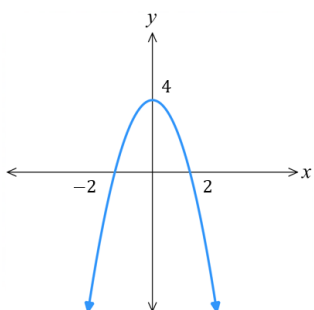
$$f(-x) = -x(-x - 4) = x(x + 4)$$

$$-f(-x) = -x(x + 4)$$

- c Hence sketch the graph after reflection.



- 4 For the graph of $f(x) = 4 - x^2$,
 a Sketch the graph,
 showing intercepts and vertex.



- b find the equation of the graph after:

180 degree rotation

reflection across x axis

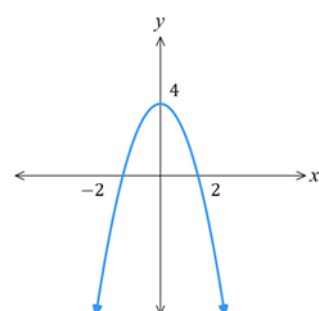
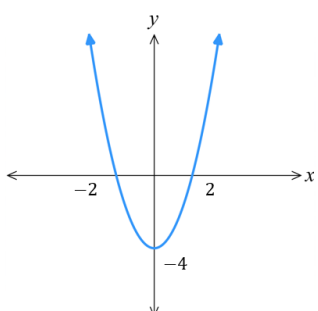
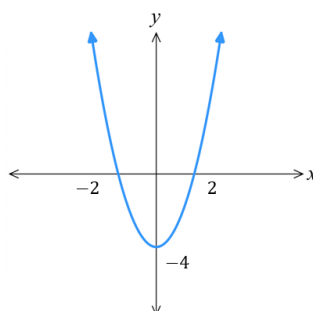
reflection across y axis

$$-f(-x) = x^2 - 4$$

$$-f(x) = x^2 - 4$$

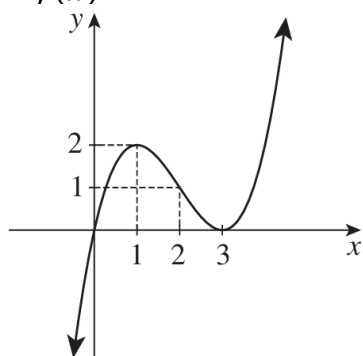
$$f(-x) = 4 - x^2$$

- c Hence sketch the graph after reflection.

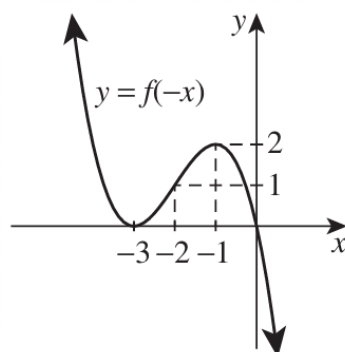


5 The graph of a function is given. Sketch the required reflection.

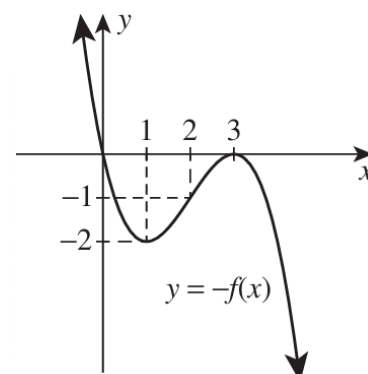
a $f(x)$



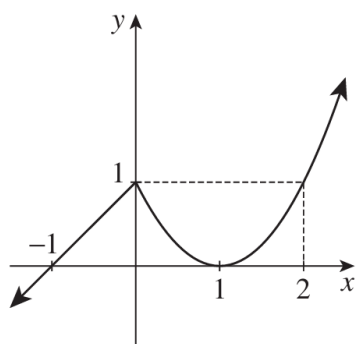
$f(-x)$



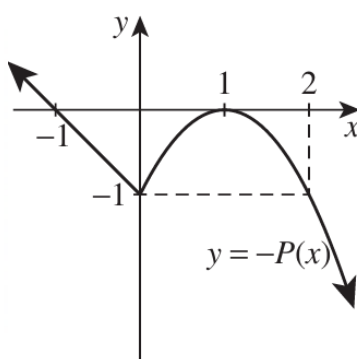
$-f(x)$



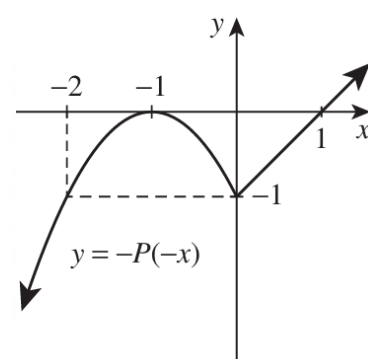
b $P(x)$



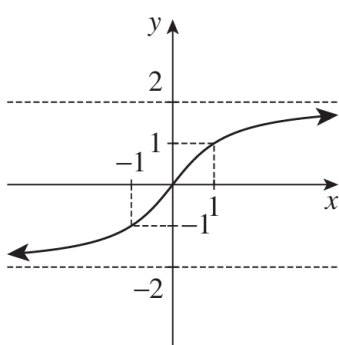
$-P(x)$



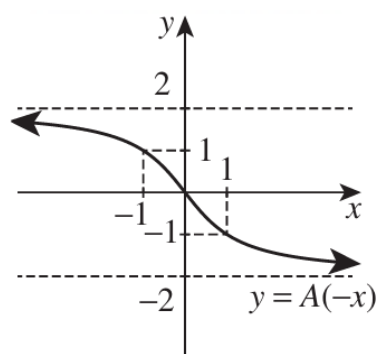
$-P(-x)$



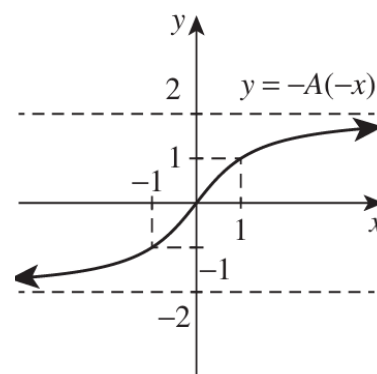
c $A(x)$



$A(-x)$



$-A(-x)$



- 6 For the function $f(x) = 2x^2$, explain why the graph of $f(-x)$ and $f(x)$ will look the same.

$$f(-x) = 2(-x)^2 = 2x^2 = f(x)$$

This is an even function.

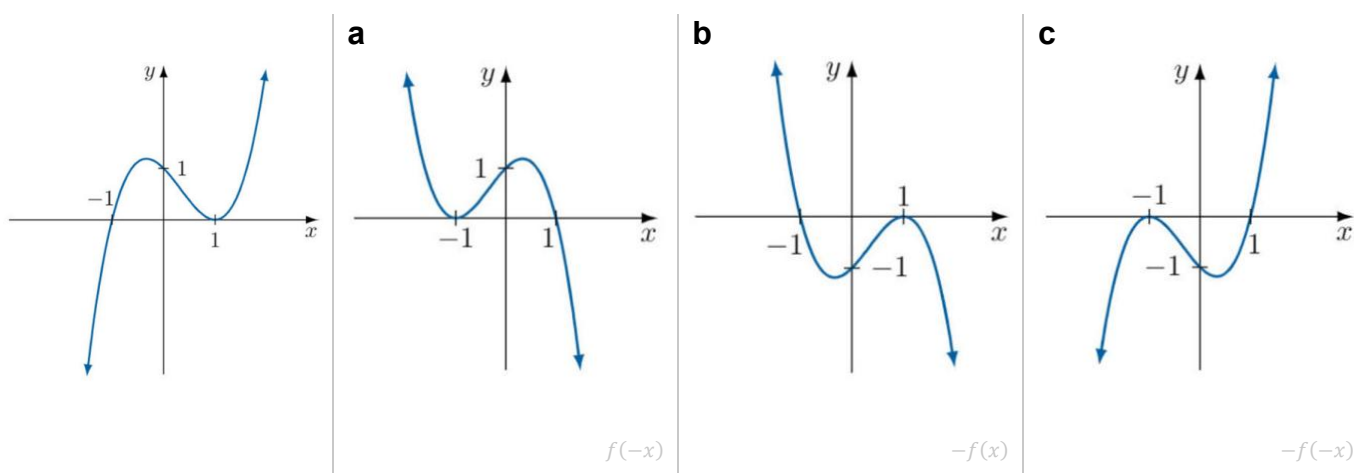
- 7 For the function $f(x) = -x^3$, explain why the graph of $f(-x)$ and $-f(x)$ will look the same.

$$f(-x) = -(-x)^3 = x^3 = -f(x)$$

This is an odd function.

- 8 The leftmost diagram shows a sketch of $y = f(x)$.

Determine the transformation $y = -f(x)$, $y = f(-x)$, $y = -f(-x)$ for each of these graphs.



- 9 Consider $f(x) = (x - 5)^2$, show that $g(x) = (x + 5)^2$ is a reflection across the y axis using function notation.

$$f(-x) = (-x - 5)^2 = (x + 5)^2 = g(x)$$

- 10 Consider the function $y = |x - 2|$

- a Write down the equation of the reflection across the x -axis.

$$-f(x) = -|x - 2|$$

- b Write down the reflection of the curve from part a across the y -axis.

$$-f(-x) = -|-x - 2| = -|x + 2|$$

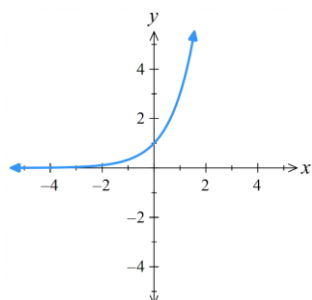
- c Verify you get the same result by reflecting $y = |x - 2|$ across the y -axis then x -axis.

$$f(-x) = |-x - 2| = |x + 2|$$

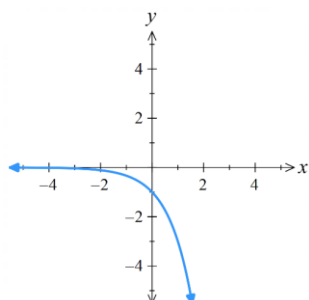
$$-f(x) = -|x + 2|$$

11 For each equation, determine whether the subsequent equation is a reflection or a rotation of the first equation, then sketch each graph.

a $y = 3^x$

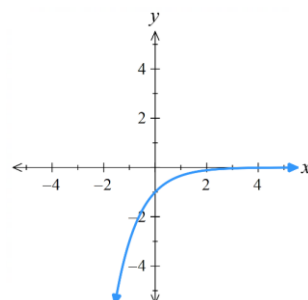


$y = -3^x$



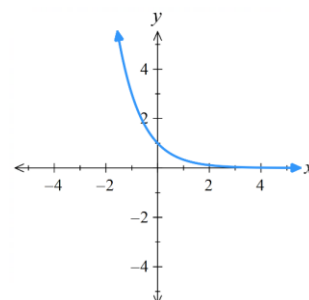
$-f(x)$, reflection across x

$y = -3^{-x}$



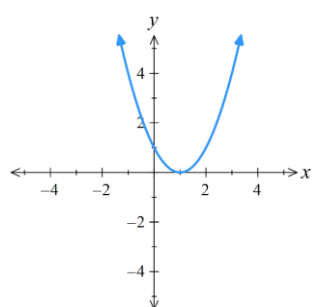
$-f(-x)$, 180 deg rotation

$y = 3^{-x}$

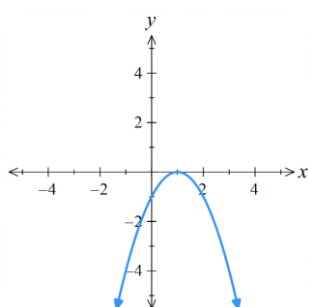


$f(-x)$, reflection across y

b $y = (x - 1)^2$

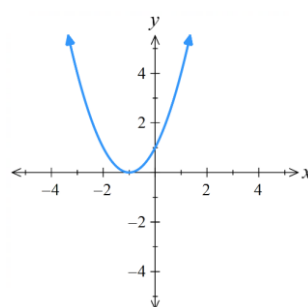


$y = -(x - 1)^2$



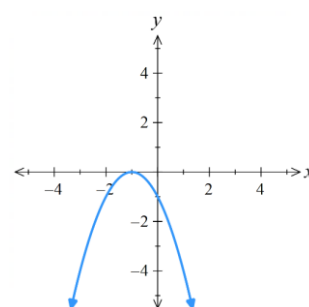
$-f(x)$, reflection across x

$y = (x + 1)^2$



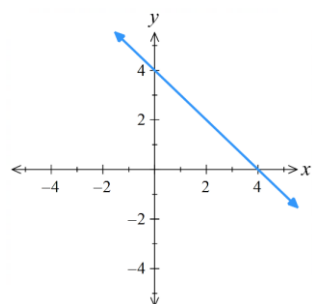
$f(-x)$, reflection across y

$y = -(x + 1)^2$

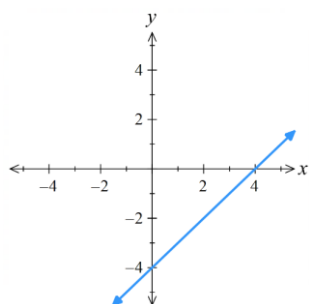


$-f(-x)$, 180 deg rotation

c $y = 4 - x$

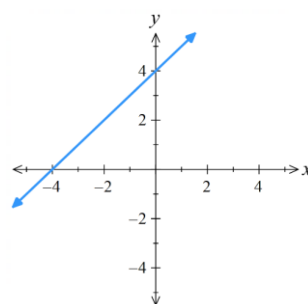


$y = x - 4$



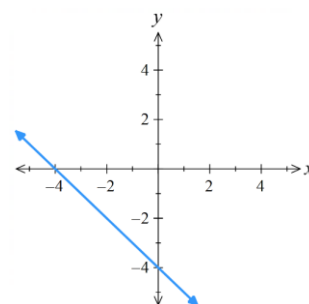
$-f(x)$, reflection across x

$y = x + 4$



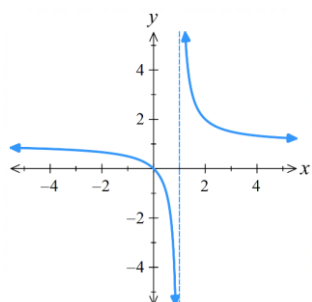
$f(-x)$, reflection across y

$y = -x - 4$

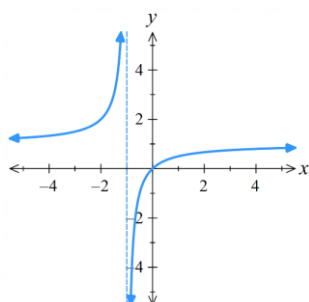


$-f(-x)$, 180 deg rotation

d $y = 1 + \frac{1}{x-1}$

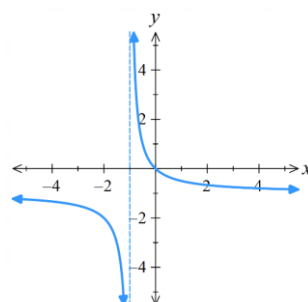


$y = 1 - \frac{1}{x+1}$



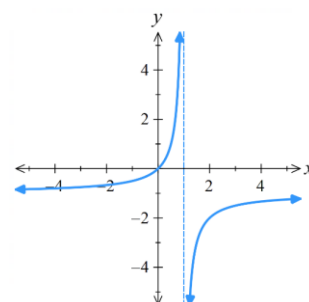
$f(-x)$, reflection across y

$y = -1 + \frac{1}{x+1}$



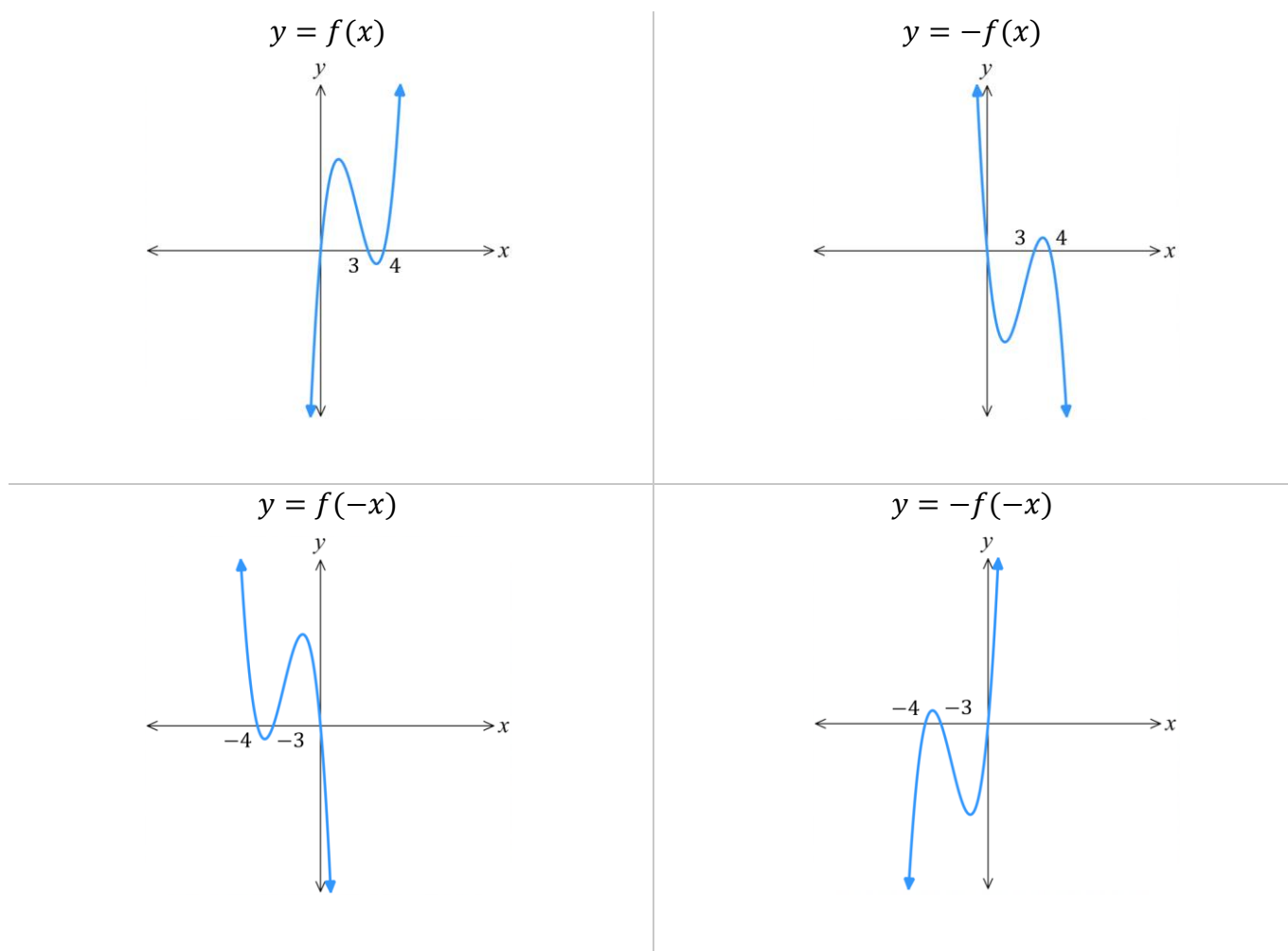
$-f(-x)$, 180 deg rotation

$y = -1 + \frac{1}{1-x}$



$-f(x)$, reflection across x

- 12** $f(x) = x^3 - 7x^2 + 12x$. Sketch the graph of $y = f(x)$, $y = -f(x)$, $y = f(-x)$ and $y = -f(-x)$.
Note: you do not need to find the coordinates of the turning point.



- 13** Which of the following equations is the rotation of $y = (x - 1)(x + 1)^2$ by 180° ?

$$y = (x^2 - 1)(x + 1)$$

$$y = (x^2 - 1)(x - 1)$$

$$y = (x^2 - 1)(1 - x)$$

$$\begin{aligned} \text{Let } f(x) &= (x - 1)(x + 1)^2 \\ f(-x) &= (-x - 1)(-x + 1)^2 \\ -f(-x) &= -(-x - 1)(1 - x)^2 \\ &= (x + 1)(1 - x)^2 \\ &= (x + 1)(1 - x)(1 - x) \\ &= (1 - x^2)(1 - x) \\ &= -(x^2 - 1) \times -(x - 1) \\ &= (x^2 - 1)(x - 1) \end{aligned}$$

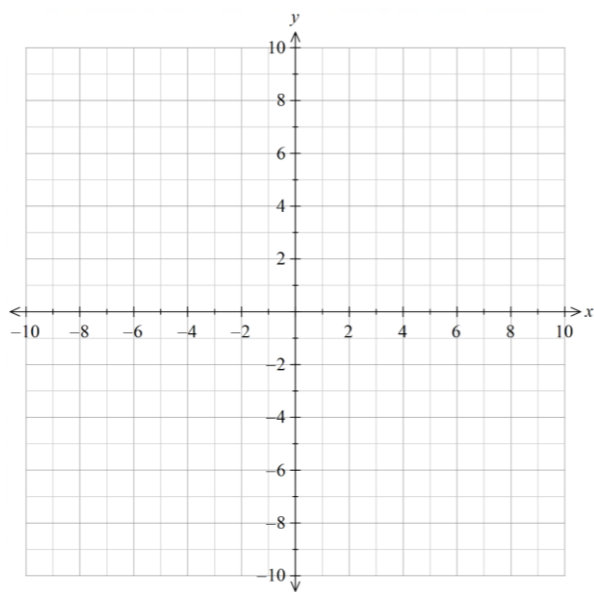
DILATIONS OF FUNCTIONS

Investigation Dilations of Functions.

Consider the function $f(x) = x^2 - x$. Write the equation and graph using a table of values:

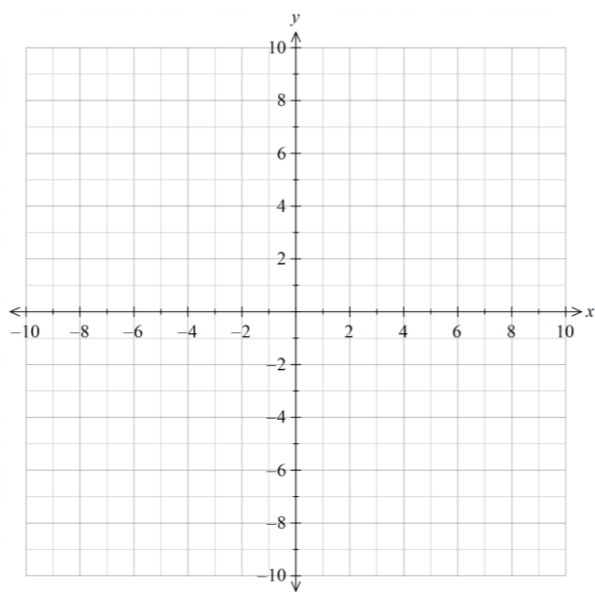
$y = f(x) = \dots\dots\dots$

x	-3	-2	-1	0	1	2	3
y							



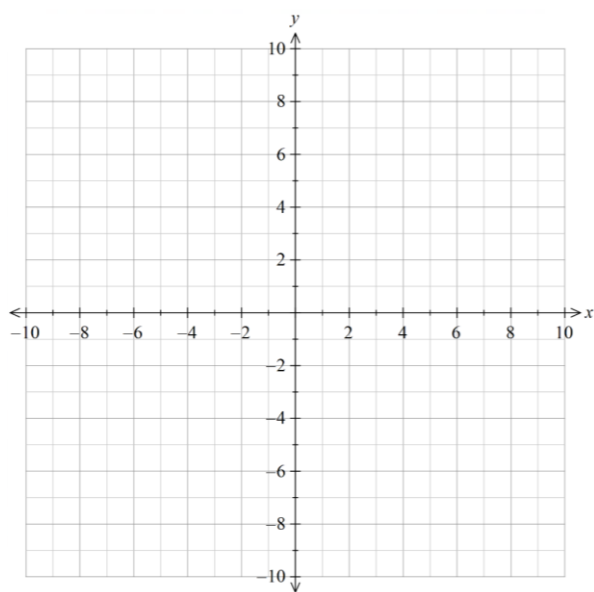
$y = 2f(x) = \dots\dots\dots$

x	-3	-2	-1	0	1	2	3
y							



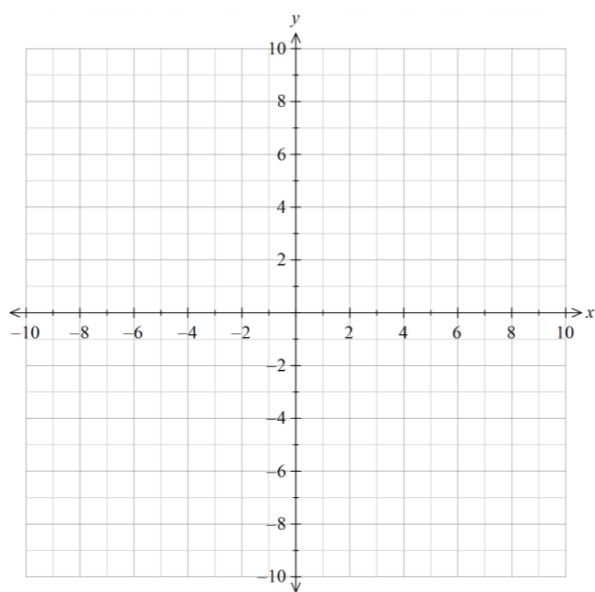
$y = \frac{1}{2}f(x) = \dots\dots\dots$

x	-3	-2	-1	0	1	2	3
y							



$y = f(2x) = \dots\dots\dots$

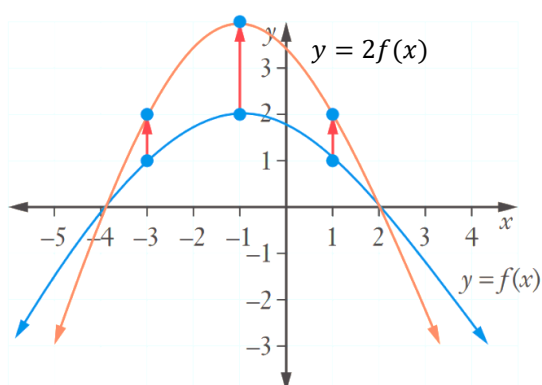
x	-3	-2	-1	0	1	2	3
y							



Vertical dilations $y = kf(x)$	Horizontal dilations $y = f(ax)$
All y values are multiplied by a factor of k We call this a dilation of factor k $k > 1$, graph is stretched away from x-axis $0 < k < 1$, graph compressed towards x-axis	All x values are replaced with ax We call this a dilation of factor $\frac{1}{a}$ $a > 1$, graph is compressed towards y-axis $0 < a < 1$, graph stretched away from y-axis

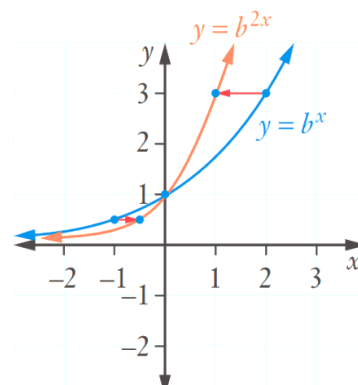
Example Sketch graphs using dilations.

The graph shown is $y = f(x)$.
Sketch the graph $y = 2f(x)$.



We dilated v by factor,
meaning we multiplied every y value by

The graph shown is $y = f(x)$.
Sketch the graph $y = f(2x)$.



We dilated by factor,
meaning we multiplied every x value by

Key ideas

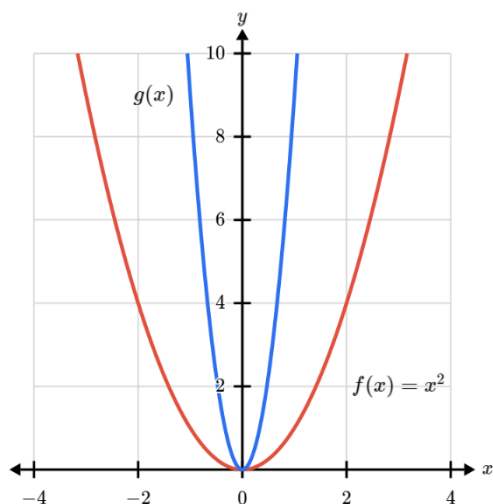
- Let $f(x)$ be a function and $a, k > 0$,
 - $y = kf(x)$ is stretched | compressed v..... if $k > 1$
 - $y = kf(x)$ is stretched | compressed v..... if $k < 1$
 - $y = f(ax)$ is stretched | compressed h..... if $a > 1$
 - $y = f(ax)$ is stretched | compressed h..... if $a < 1$
 - for $y = kf(x)$, we call this a dilation of factor, we multiply every y value by
 - for $y = f(ax)$, we call this a dilation of factor, we multiply every x value by

Check your understanding

Consider the graph of $f(x) = x^2$

The graph has been dilated **horizontally**.

What would the equation of $g(x)$ be?

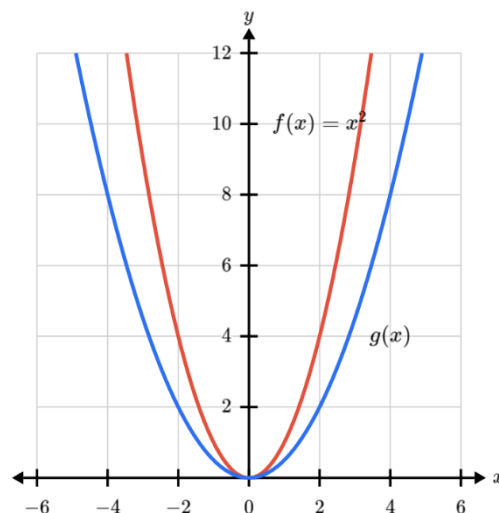


- A. $g(x) = \left(\frac{1}{3}x\right)^2$
- B. $g(x) = \frac{1}{3}x^2$
- C. $g(x) = 3x^2$
- D. $g(x) = (3x)^2$

Consider the graph of $f(x) = x^2$.

The graph has been dilated **vertically**.

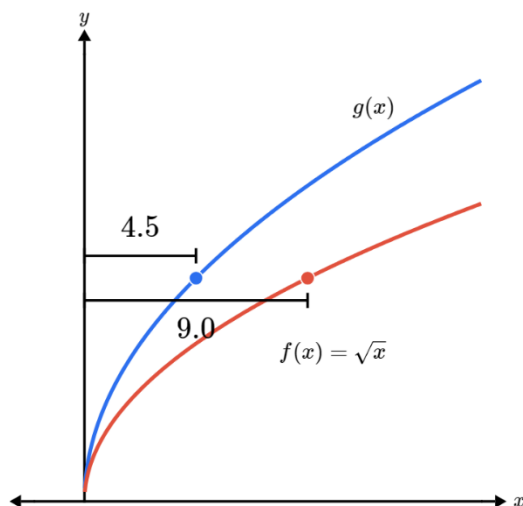
What would the equation of $g(x)$ be?



- A. $g(x) = 2x^2$
- B. $g(x) = \frac{1}{2}x^2$
- C. $g(x) = (2x)^2$
- D. $g(x) = \left(\frac{x}{2}\right)^2$

Consider the graph of $f(x) = \sqrt{x}$

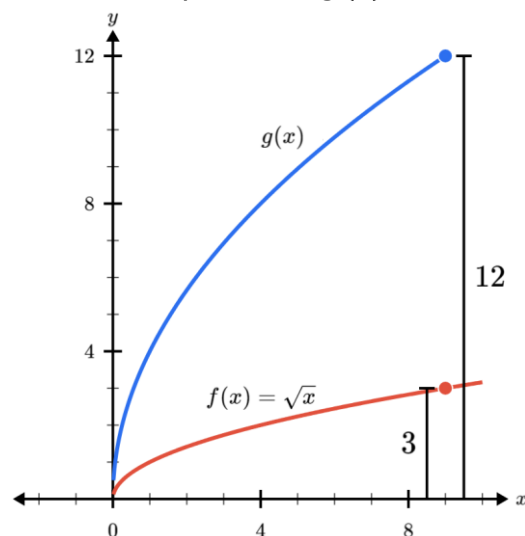
What would the equation of $g(x)$ be?



- A. $g(x) = 2\sqrt{x}$
- B. $g(x) = \sqrt{2x}$
- C. $g(x) = \sqrt{4.5x}$
- D. $g(x) = 4.5\sqrt{x}$

Consider the graph of $f(x) = \sqrt{x}$

What would the equation of $g(x)$ be?



- A. $g(x) = 3\sqrt{x}$
- B. $g(x) = 4\sqrt{x}$
- C. $g(x) = \sqrt{3x}$
- D. $g(x) = \sqrt{4x}$

1 Determine the dilation factor if the original graph is $y = f(x)$.

a $y = 2f(x)$

Vertical by factor 2

b $y = 3f(x)$

c $y = \frac{1}{3}f(x)$

d $y = \frac{2}{3}f(x)$

e $y = \frac{f(x)}{3}$

f $y = \frac{2f(x)}{3}$

g $y = f(2x)$

Horizontal by factor $\frac{1}{2}$

h $y = f(3x)$

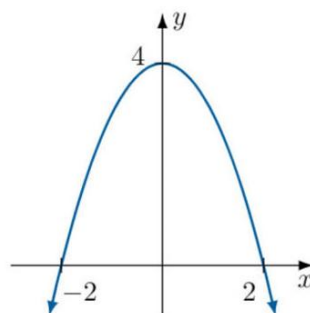
i $y = f\left(\frac{x}{2}\right)$

j $y = f\left(\frac{x}{3}\right)$

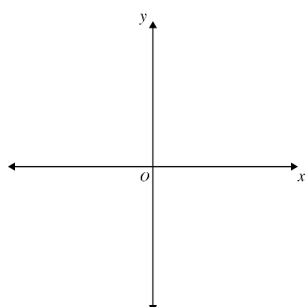
k $y = f\left(\frac{2x}{3}\right)$

l $y = f\left(\frac{3x}{2}\right)$

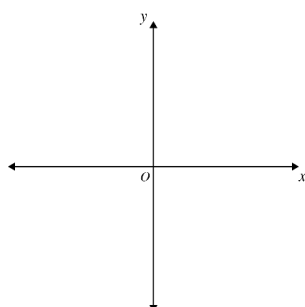
2 The diagram shows the graph of $y = f(x)$
Describe the transformation and sketch the graph of:



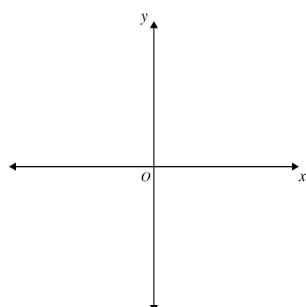
a $y = 2f(x)$



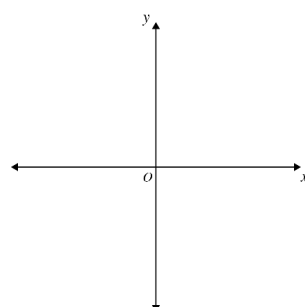
b $y = \frac{1}{2}f(x)$



c $y = f(2x)$

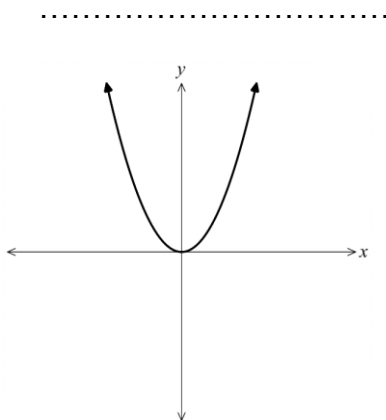


d $y = f\left(\frac{x}{2}\right)$

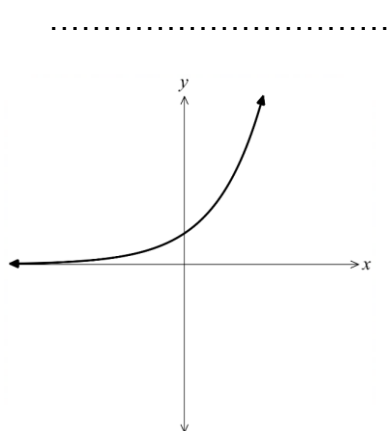


- 3** Write down the new equation for each function or relation after the given dilation.
Draw a sketch of the image after the dilation.

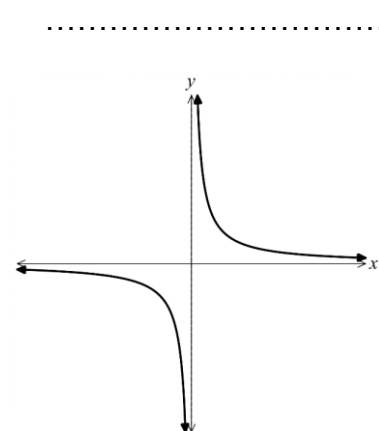
a $y = x^2$
horizontally by $\frac{1}{2}$



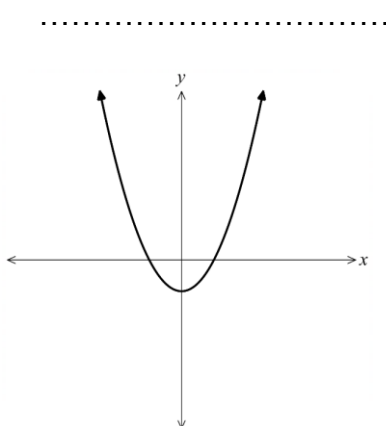
b $y = 2^x$
vertically by 2



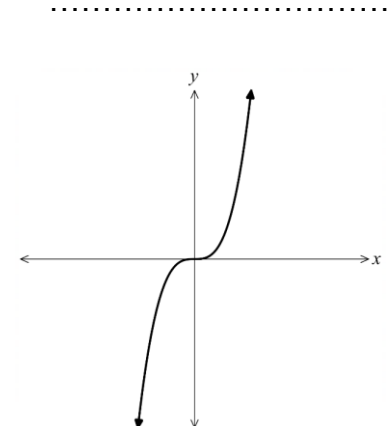
c $y = \frac{1}{x}$
vertically by 2



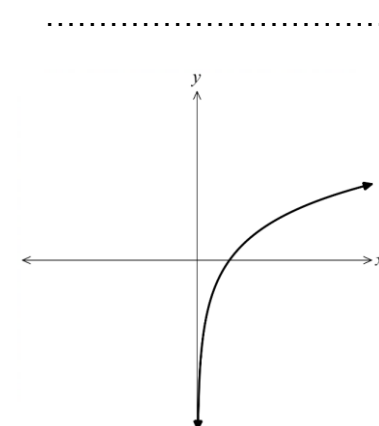
d $y = x^2 - 1$
horizontally by 2



e $y = x^3$
vertically by $\frac{1}{3}$



f $y = \log_2 x$
horizontally by 2



- 4** Consider $y = x^2$ and $\frac{y}{3} = x^2$. Explain why the second graph is a vertical dilation of factor 3.

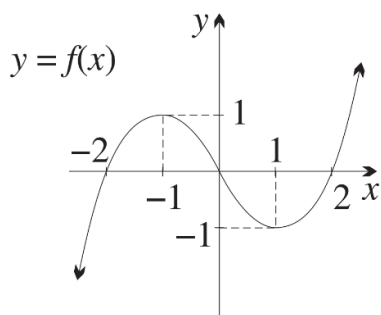
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- 5** Consider $f(x) = x(2 + x)$.
Explain why $g(x) = 2x(2 + 2x)$ is a horizontal dilation, not a vertical dilation.

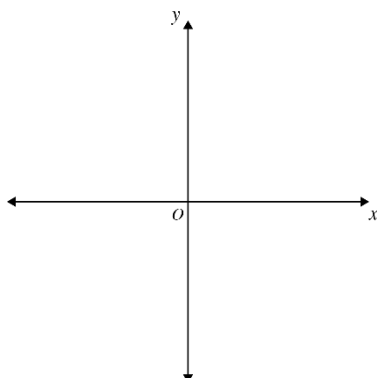
.....

- 6** In each case an unknown function has been drawn.
Use dilations to draw the new functions indicated below it.

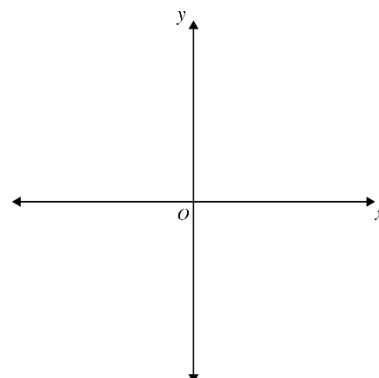
a $y = f(x)$



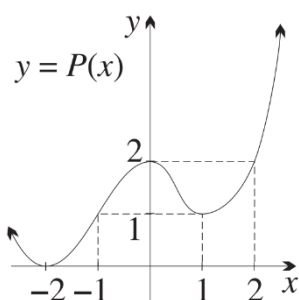
$y = 2f(x)$



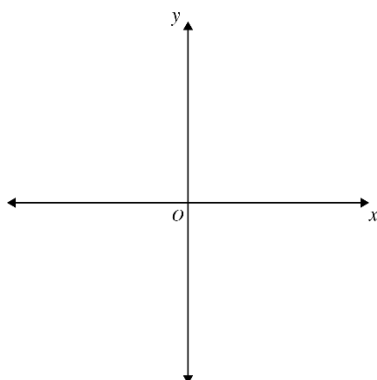
$y = f(2x)$



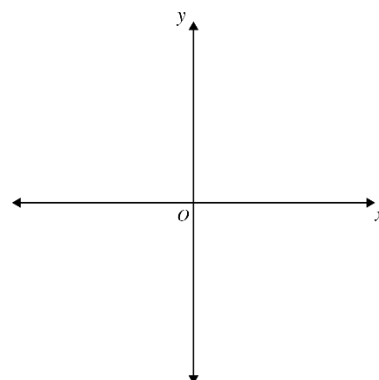
b $y = P(x)$



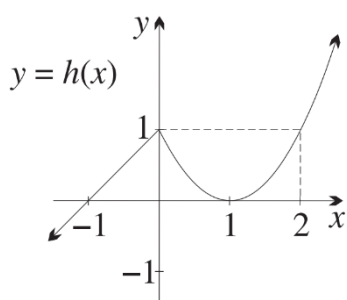
$y = \frac{1}{2}P(x)$



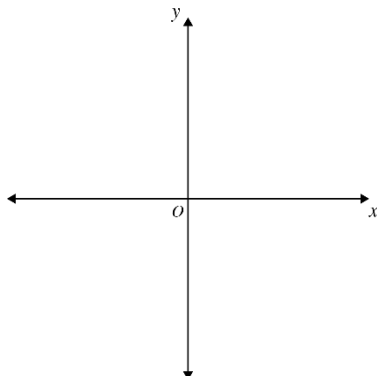
$y = P\left(\frac{x}{2}\right)$



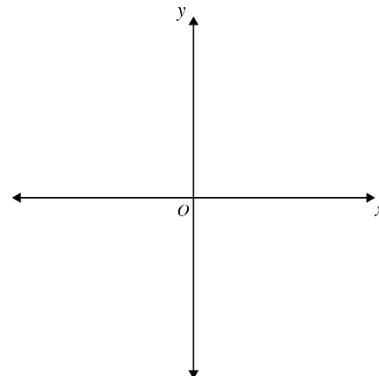
c $y = h(x)$



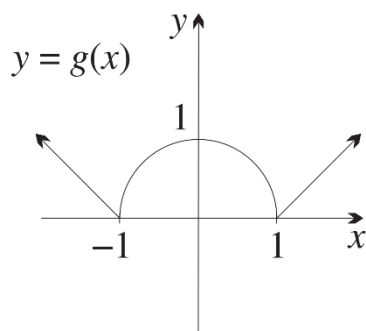
$\frac{y}{2} = h(x)$



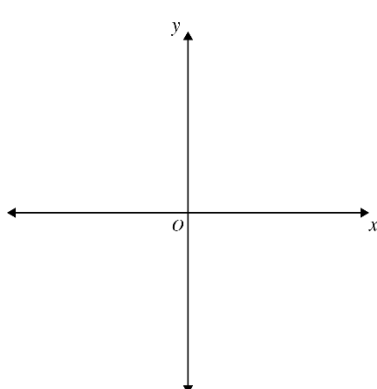
$y = h\left(\frac{x}{2}\right)$



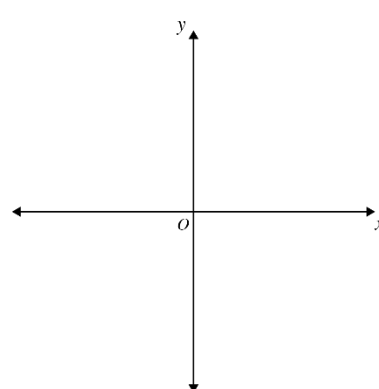
d $y = g(x)$



$2y = g(x)$



$y = g(2x)$

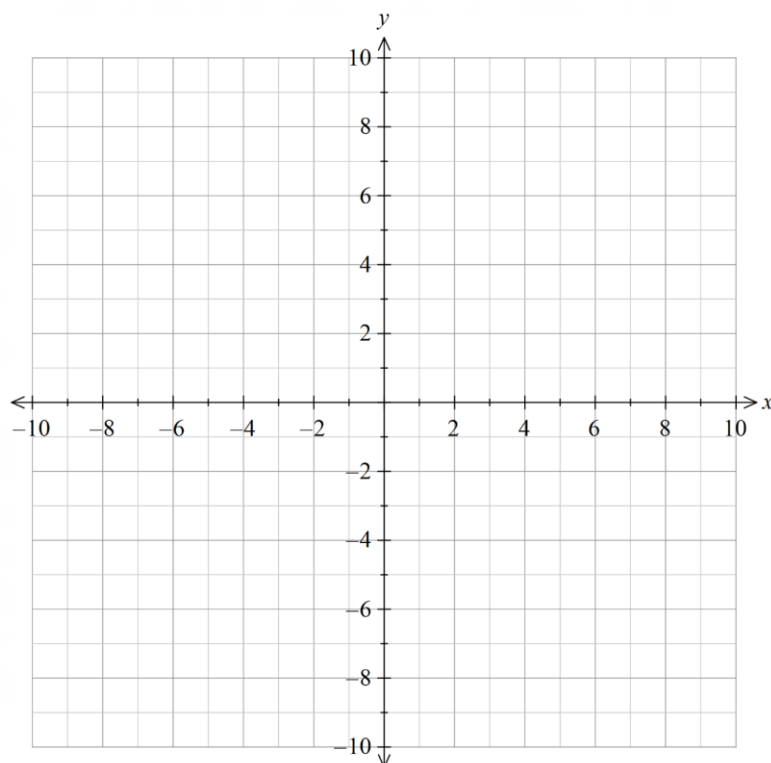


7 Graph the three given equations on one set of axes by using dilations.

a $y = x(2 + x)$

b $y = 2x(2 + x)$

c $y = \frac{x}{2} \left(2 + \frac{x}{2} \right)$

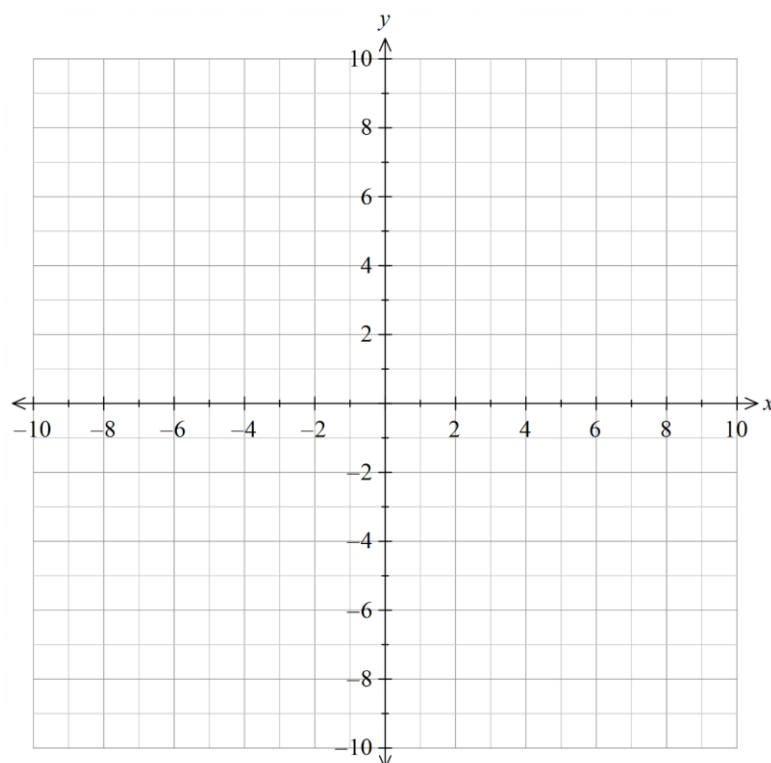


8 Graph the three given equations on one set of axes by using dilations

a $x^2 + y^2 = 4$

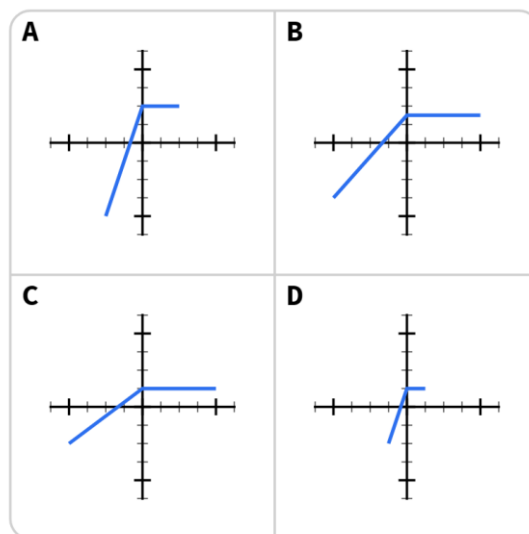
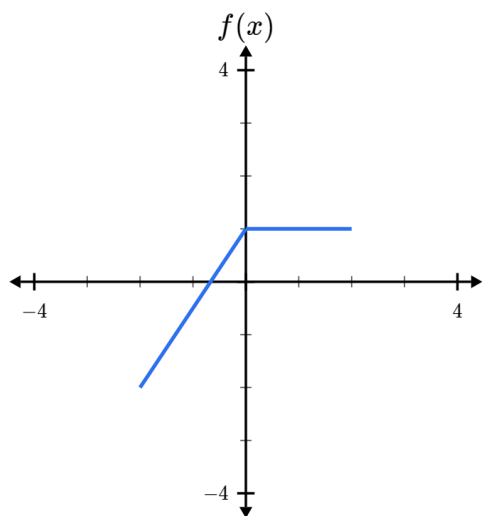
b $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 4$

c $(2x)^2 + (3y)^2 = 4$



9 Consider the graph of $f(x)$ below.

Which is the graph of the transformed function $\frac{3}{2}f\left(\frac{1}{2}x\right)$?



10 Consider the parabola $y = x^2$

a Write the equation of $g(x) = 4f(x)$. What dilation is this?

.....

b Write the equation of $h(x) = f(2x)$. What dilation is this?

.....

c What do you notice about the answers to parts a and b?

.....

11 Consider the hyperbola $y = \frac{1}{x}$

a Write the equation of $g(x) = 2f(x)$. What dilation is this?

.....

b Write the equation of $h(x) = f\left(\frac{x}{2}\right)$. What dilation is this?

.....

c What do you notice about the answers to parts a and b?

12 In each case identify how the graph of the second equation can be obtained from the graph of the first by a suitable dilation.

e $y = x^2 - 2x$ and $y = 3x^2 - 6x$

f $y = \frac{1}{x-4}$ and $y = \frac{1}{2x-4}$

g $y = \cos x$ and $y = \cos \frac{x}{4}$

h $y = \frac{1}{x+1}$ and $y = \frac{2}{x+1}$

- 13 Consider the parabola $f(x) = x^2$. $g(x) = 5x^2$.

Lara says “ $g(x)$ is the graph of $f(x)$ stretched away from the x -axis by factor of 5”

George says “ $g(x)$ is the graph of $f(x)$ compressed towards the y -axis by factor of $\frac{1}{\sqrt{5}}$ ”

Explain why they are both correct.

.....

.....

14

- a Explain why the graph of $g(x) = (2x - 4)^2$ is shifted horizontally by 2 and NOT 4.

.....

.....

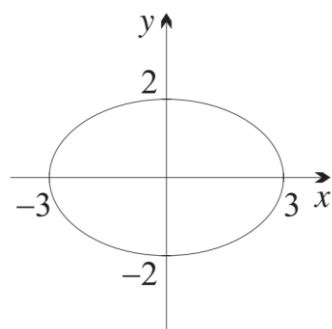
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- b Where's the turning point of the graph of $g(x)$?

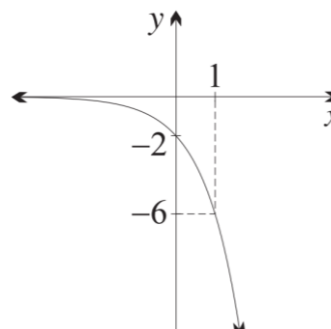
.....

- 15 Describe each graph below as the standard curve transformed by dilations and write its equation.

- a Circle $x^2 + y^2 = r^2$



- b Exponential $y = a^x$



16 The mass M grams of a certain radioactive substance after t years is modelled by the formula

$$M = 3 \times 2^{-\frac{1}{53}t}$$

- a Find the initial mass.
- b Find the time taken for the mass to halve, this is called the half-life.
.....
.....
- c Suppose the initial mass is doubled.
 - i Explain this in terms of a dilation and hence write down the new equation for M .
.....
 - ii Show that the half-life is unchanged.
.....
.....

17 Show that horizontally translating $y = e^x$ is equivalent to dilating it vertically.
.....
.....
.....

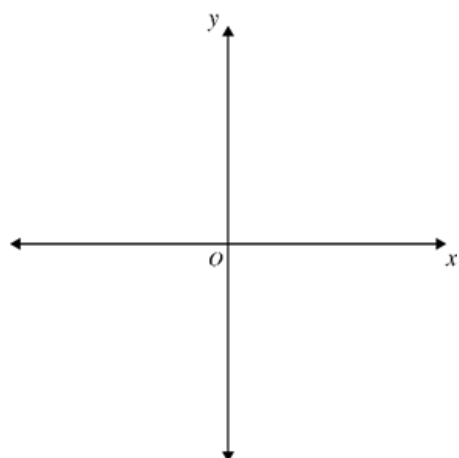
18 Show that vertically translating $y = x^2$ is equivalent to horizontally dilating it.
.....
.....
.....

19 Show that horizontally dilating $y = \log_b x$ is equivalent to translating it vertically.
.....
.....
.....

QUICKDRAW MCGRAW

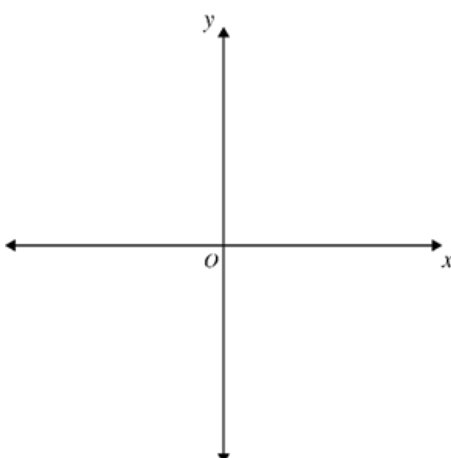
Reflect about x -axis.

New equation:



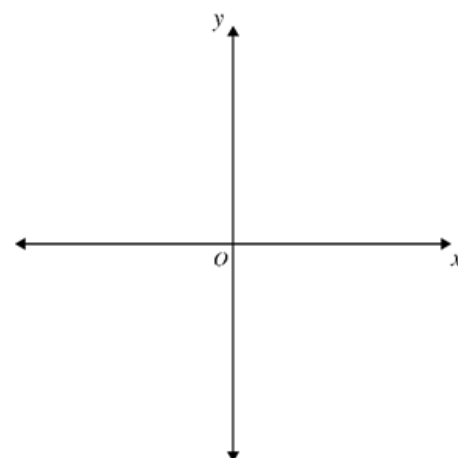
Shift up by 2 units.

New equation:



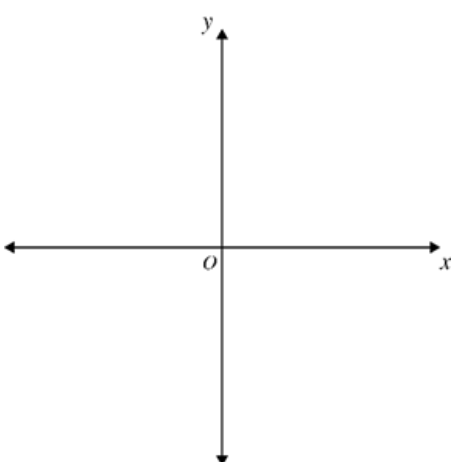
Reflect about y -axis.

New equation:

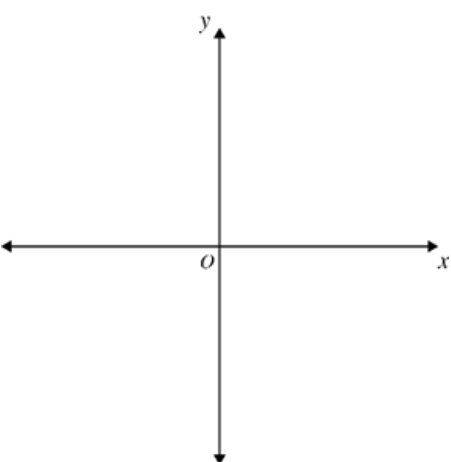


Shift left by 2 units.

New equation:

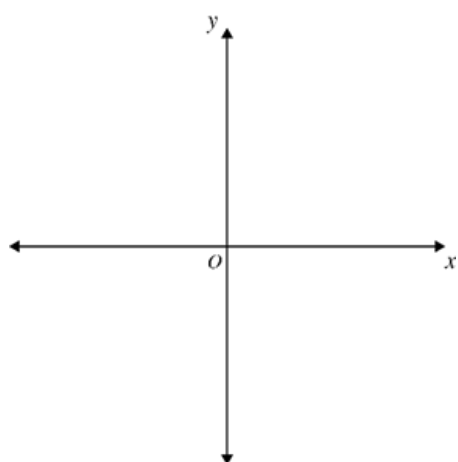


Original Equation:



Shift right by 2 units.

New equation:

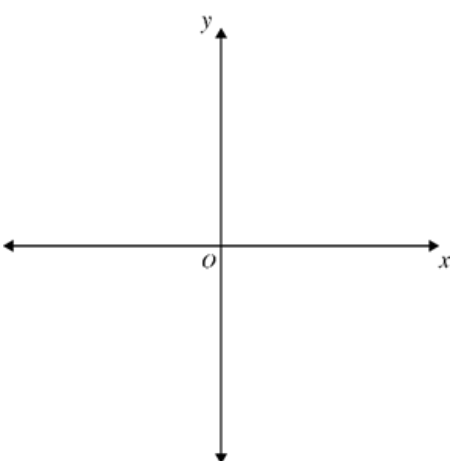


**QUICK
DRAW
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Shift down by 2 units.

New equation:



Rotate by 180°

New equation:

