Student Name

Mathematics Stage 5 Paths

POLYNOMIALS

Book 2 Divide polynomials

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CONTENTS

OVERVIEW

SYLLABUS CONTENT

MA5-POL-P-01 defines, operates with and graphs polynomials and applies the factor and remainder theorems to solve problems

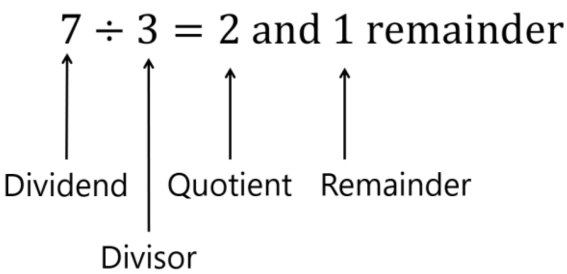
Divide polynomials

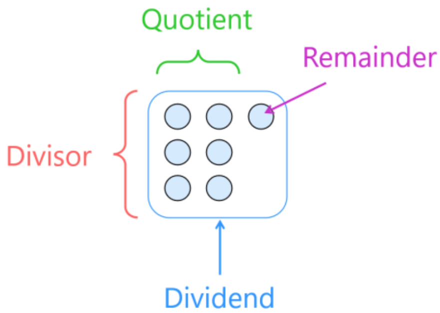
- Identify the dividend, divisor, quotient and remainder in numerical division
- Divide a polynomial by a linear polynomial to find the quotient and remainder
- Express a polynomial in the form $P(x) = D(x)Q(x) + R(x)$, where $D(x)$ is the divisor, $Q(x)$ is the quotient and $R(x)$ is the remainder

LONG DIVISION

LANGUAGE OF DIVISION

- Division involves finding how many times a number goes into another number, and the number that is left over.
- We have special terms for these four numbers to make communication clear.





Example Identify numbers in a division question.

$18 \div 3 = 6$	$128 \div 9 = 14 \text{ r } 2$
18: Dividend 3: Divisor 6: Quotient	Dividend: 128 Divisor 9 Quotient: 14 Remainder: 2

Identify the dividend, divisor, quotient, and remainder in these division questions.

a $56 \div 7 = 8$ 56: dividend 7: divisor 8: quotient	**b** $63 \div 3 = 20 \text{ r } 2$ 3: divisor 63: dividend 20: quotient 2: remainder	**c** $39 \div 5 = 7 \text{ r } 4$ 4: remainder 7: quotient 39: dividend 5: divisor
d $51 \div 17 = 3$ Divisor: 17 Quotient: 3 Dividend: 51 Remainder: 0	**e** $59 \div 8 = 7 \text{ r } 3$ Dividend: 59 Remainder: 3 Divisor: 8 Remainder: 3	**f** $78 \div 9 = 8 \text{ r } 2$ Remainder: 2 Quotient: 8 Dividend: 78 Divisor: 9

LONG DIVISION

- Polynomial division requires knowledge of long division. We will review that now.

Example Divide two integers using long division.

$405 \div 7$

Step 1

$$7\overline{)405}$$

Step 2

$$\begin{array}{r} 5 \\ 7\overline{)405} \\ \underline{350} \\ 55 \end{array}$$

Step 3

$$\begin{array}{r} 57 \text{ r } 6 \\ 7\overline{)405} \\ \underline{350} \\ 55 \\ \underline{49} \\ 6 \end{array}$$

$832 \div 3$

Step 1

$$3\overline{)832}$$

Step 2

$$\begin{array}{r} 2 \\ 3\overline{)832} \\ \underline{600} \\ 232 \end{array}$$

Step 3

$$\begin{array}{r} 27 \\ 3\overline{)832} \\ \underline{600} \\ 232 \\ \underline{210} \\ 22 \end{array}$$

Step 4

$$\begin{array}{r} 277 \text{ r } 1 \\ 3\overline{)832} \\ \underline{600} \\ 232 \\ \underline{210} \\ 22 \\ \underline{21} \\ 1 \end{array}$$

Complete these divisions using the long division algorithm.

a $598 \div 4$

147 r 1

b $2178 \div 7$

311 r 1

c $1324 \div 12$

110 r 4

d $5249 \div 7$

749 r 6

POLYNOMIAL DIVISION

remainder

dividend $\rightarrow \dfrac{x^3 - x^2 + x - 1}{x + 2} = x^2 - 3x + 7 - \dfrac{15}{x+2}$

divisor quotient

- At each step, **divide the leading term** of the remainder **by the leading term** of the divisor.
- We write the result in the following form, with dividend $P(x)$, divisor $D(x)$, quotient $Q(x)$, and remainder $R(x)$:

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

Example Divide a polynomial by a linear polynomial.

$(3x^2 - 11x + 2) \div (x - 1) = 3x - 8 + \frac{-6}{x-1}$

Step 1

$$x - 1\overline{)\,3x^2 - 11x + 2}$$

Step 2

$$\begin{array}{r} 3x \phantom{{}-11x+2} \\ x - 1\overline{)\,3x^2 - 11x + 2} \\ -\ \underline{(3x^2 - 3x)\phantom{{}+2}} \\ -8x + 2 \end{array}$$

Step 3

$$\begin{array}{r} 3x - 8 \phantom{{}+2} \\ x - 1\overline{)\,3x^2 - 11x + 2} \\ -\ \underline{(3x^2 - 3x)\phantom{{}+2}} \\ -8x + 2 \\ -\ \underline{(-8x + 8)} \\ -6 \end{array}$$

$(8x^2 + 4x^2 + 12x - 5) \div (2x + 1) = 4x^2 + 6 + \frac{-11}{2x+1}$

Step 1

$$2x + 1\overline{)\,8x^3 + 4x^2 + 12x - 5}$$

Step 2

$$\begin{array}{r} 4x^2 \phantom{{}+12x-5} \\ 2x + 1\overline{)\,8x^3 + 4x^2 + 12x - 5} \\ -\ \underline{(8x^3 + 4x^2)\phantom{{}+12x-5}} \\ 12x - 5 \end{array}$$

Step 3

$$\begin{array}{r} 4x^2 \phantom{{}+12x} + 6 \\ 2x + 1\overline{)\,8x^3 + 4x^2 + 12x - 5} \\ -\ \underline{(8x^3 + 4x^2)\phantom{{}+12x-5}} \\ 12x - 5 \\ -\ \underline{(12x + 6)} \\ -11 \end{array}$$

Divide these polynomials. Express your answer in the form:

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

a $(x^2 - 5x + 3) \div (x + 1)$

$(x-6) + \frac{9}{x+1}$

b $(x^2 - 2x + 5) \div (x - 4)$

$x + 2 + \frac{13}{x-4}$

c $(x^3 - x^2 - 17x + 24) \div (x - 4)$

$x^2 + 3x - 5 + \frac{4}{x-4}$

d $(4x^3 - 4x^2 + 7x + 14) \div (2x + 1)$

$2x^2 - 3x + 5 + \frac{9}{2x+1}$

1 Complete these divisions.

a Example

$$
\begin{array}{r}
x + 5 \\
x+4\,\overline{)\,x^2 + 9x + 18} \\
-(x^2 + 4x) \\
\hline
5x + 18 \\
-(5x + 20) \\
\hline
-2
\end{array}
$$

$$\frac{x^2 + 9x + 18}{x + 4} = (x + 5) + \frac{-2}{x + 4}$$

b

$$
\begin{array}{r}
x + \square \\
x+1\,\overline{)\,x^2 + 9x + 18} \\
-(x^2 + x) \\
\hline
8x + 18 \\
-(\square + \square) \\
\hline
10
\end{array}
$$

$$\frac{x^2 + 9x + 18}{x + 1} = (\qquad) + \frac{10}{(\qquad)}$$

$(x + 8) + \frac{10}{x + 1}$

c

$$
\begin{array}{r}
x + \square \\
x+3\,\overline{)\,x^2 + 9x + 18} \\
-(x^2 + \square) \\
\hline
\square + 18 \\
-(\square + \square) \\
\hline
\square
\end{array}
$$

$x + 6$

d

$$
\begin{array}{r}
x + \square \\
x-4\,\overline{)\,x^2 + 9x + 18} \\
-(x^2 \square) \\
\hline
\square + \square \\
-(\square - \square) \\
\hline
\square
\end{array}
$$

$(x + 13) + \frac{70}{x + 4}$

e

$$
\begin{array}{r}
\square - \square \\
x+1\,\overline{)\,5x^2 - 2x + 3} \\
-(5x^2 + 5x) \\
\hline
\square + \square \\
-(\square - 7) \\
\hline
\square
\end{array}
$$

$(5x - 7) + \frac{10}{x + 1}$

f

$$
\begin{array}{r}
2x^2 + \square - \square \\
3x+2\,\overline{)\,6x^3 + 13x^2 - 12x + 8} \\
-(6x^3 + \square) \\
\hline
9x^2 - 12x \\
-(9x^2 \square) \\
\hline
-18x \square \\
-(\square) \\
\hline
20
\end{array}
$$

$(2x^2 + 3x - 6) + \frac{20}{3x + 2}$

g

$$
\begin{array}{r}
x^3 + \square + \square \\
x-3\,\overline{)\,x^4 - x^3 - 2x^2 - 12x}
\end{array}
$$

$x^3 + 2x^2 + 4x$

2 Perform the following divisions and express each result in this form

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

a $(x^2 - 4x + 1) \div (x + 1)$

$(x-5) + \dfrac{6}{x+1}$

b $(x^2 - 6x + 5) \div (x - 5)$

$x - 1$

c $(2x^3 + 2x^2 - x - 3) \div (x + 2)$

$(2x^2 - 2x + 3) - \dfrac{9}{x+2}$

d $(-x^3 + x^2 - 10x + 4) \div (x - 4)$

$(-x^2 - 3x - 22) - \dfrac{84}{x-4}$

e $\dfrac{4x^3 - 4x^2 + 7x + 14}{2x + 1}$

$(2x^2 - 3x + 5) + \dfrac{9}{2x+1}$

f $\dfrac{x^4 + x^3 - x^2 - 5x - 3}{x - 1}$

$(x^3 + 2x^2 + x - 4) - \dfrac{7}{x-1}$

3 Consider the example $(4x^3 + x - 550) \div (x - 5)$
This polynomial is missing the x^2 term. When there are “gaps” in a polynomial, it is useful to insert any missing terms using a coefficient of 0 so you can keep the terms lined up.

$$\begin{array}{r} \boxed{} + 20x + \boxed{} \\ x - 5 \,\big)\overline{\,4x^3 + 0x^2 + x - 550} \end{array}$$

$(4x^2 + 2x + 101) + \dfrac{450}{x - 5}$

Perform the following divisions and express each result as $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$

a $(x^3 - 5x + 3) \div (x - 2)$

$(x^2 + 2x - 1) + \dfrac{1}{x - 2}$

b $(2x^3 + x^2 - 11) \div (x + 1)$

$(2x^2 - x + 1) - \dfrac{12}{x + 1}$

c $\dfrac{2x^3 - 3}{2x - 4}$

$(x^2 + 2x + 4) + \dfrac{13}{2x - 4}$

d $\dfrac{3x^4 - 4x^3 + 4x - 9}{x - 2}$

$(3x^3 + 2x^2 + 4x + 12) + 15$

4 Divide these polynomials and express in the form $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$

a $(-2x^3 - 2x^2 - 5x + 7) \div (x + 4)$

$-2x^2 + 6x - 29 + \frac{123}{x+4}$

b $(-5x^3 + 11x^2 - 2x - 20) \div (x - 3)$

$-5x^2 - 4x - 14 + \frac{-62}{x-3}$

c $(6x^4 - 5x^3 + 9x^2 - 8x + 2) \div (2x - 1)$

$3x^2 - x^2 + 4x - 2$

d $(10x^4 - x^3 + 3x^2 - 3x - 2) \div (5x + 2)$

$2x^3 - x^2 + x - 1$

e $(x^3 - 5x + 3) \div (x - 2)$

$x^2 + 2x - 1 + \frac{1}{x-2}$

f $(x^3 - 3x - 2) \div (x + 1)$

$x^2 - x - 2$

5 Some more complex questions may require you to divide by a quadratic divisor.
Look at the example and complete the divisions.

Example Divide by a polynomial by a quadratic factor.

$(3x^4 - 4x^3 + 4x - 8) \div (x^2 - 2)$

Step 1

$$
\begin{array}{r|rrrrr}
x^2-2 & 3x^4 & -4x^3 & & +4x & -8
\end{array}
$$

Step 2

$$
\begin{array}{rrrrrr}
 & 3x^2 & & & & \\
\cline{2-6}
x^2-2 \,\big) & 3x^4 & -4x^3 & & +4x & -8 \\
 & 3x^4 & & -6x^2 & & \\
\cline{2-6}
 & & -4x^3 & +6x^2 & +4x & -8
\end{array}
$$

Step 3

$$
\begin{array}{rrrrrr}
 & 3x^2 & -4x & & & \\
\cline{2-6}
x^2-2 \,\big) & 3x^4 & -4x^3 & & +4x & -8 \\
 & 3x^4 & & -6x^2 & & \\
\cline{2-6}
 & & -4x^3 & +6x^2 & +4x & -8 \\
 & & -4x^3 & & +8x & \\
\cline{3-6}
 & & & 6x^2 & -4x & -8
\end{array}
$$

Step 4

$$
\begin{array}{rrrrrr}
 & 3x^2 & -4x & +6 & & \\
\cline{2-6}
x^2-2 \,\big) & 3x^4 & -4x^3 & & +4x & -8 \\
 & 3x^4 & & -6x^2 & & \\
\cline{2-6}
 & & -4x^3 & +6x^2 & +4x & -8 \\
 & & -4x^3 & & +8x & \\
\cline{3-6}
 & & & 6x^2 & -4x & -8 \\
 & & & 6x^2 & & -12 \\
\cline{4-6}
 & & & & -4x & +4
\end{array}
$$

$$(3x^2 - 4x^3 + 4x - 8) = 3x^2 - 4x + 6 + \frac{-4x + 4}{x^2 - 2}$$

a $(x^3 - 3x^2 + 3x - 1) \div (x^2 + 5)$

$(x - 3) + \frac{-2x + 14}{x^2 + 5}$

b $(2x^3 + 4x^2 - x + 8) \div (x^2 + 3x + 2)$

$(2x - 2) + \frac{x + 12}{x^2 + 3x + 2}$

c $(2x^4 - 5x^3 + 2x^2 + 2x - 5) \div (x^2 - 2x)$

$(2x^2 - x) + \frac{2x - 5}{x^2 - 2x}$

d $(x^5 + x^3 + 5x^2 - 6x + 15) \div (x^2 + 3)$

$x^3 - 2x + 5$

QUOTIENT – REMAINDER THEOREM

- From primary school, you would have learned that:

  If $\quad a \div b = c \text{ remainder } d$

  Then $\quad a = c \times b + d$

Investigation Inverse relationship between division and multiplication.

Fill in the missing numbers.

a	$13 \div 3 = 4$ and 1 remainder	→	$13 = \ 4 \times 3 + 1$
b	$37 \div 3 = 12$ and …… remainder	→	$37 =$ …… $\times 3 + 1$
c	$96 \div 7 =$ …… and 5 remainder	→	$93 = 13 \times$ …… $+5$
d	$104 \div 20 = 5$ and …… remainder	→	$104 =$ …… $\times$ …… $+4$
e	$134 \div 14 = 9$ and 8 remainder	→	……= …… × …… + ……
f	$287 \div 30 = 9$ and 17 remainder	→	……= …… × …… + ……
g	…… ÷……= …… and ……remainder	→	$115 = 9 \times 12 + 7$
h	…… ÷……= …… and ……remainder	→	$197 = 15 \times 13 + 2$
i	…… ÷……= …… and ……remainder	→	$23 \times 82 + 15 = 1901$

QUOTIENT – REMAINDER THEOREM FOR POLYNOMIALS

- We will apply this idea to polynomials:

  Last lesson we used polynomial division to show that, given a dividend $P(x)$, divisor $D(x)$, quotient $Q(x)$, and remainder $R(x)$:

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

- We multiply every term by $D(x)$ to derive the **quotient-remainder form of a polynomial:**

$$P(x) = Q(x)D(x) + R(x)$$

- Two important facts:
  - The degree of the remainder is always less than the divisor: $\deg R(x) < \deg D(x)$
  - When $R(x) = 0$, then $Q(x)$ and $D(x)$ are factors of $P(x)$: $P(x) = Q(x) \times D(x)$

Example Write a polynomial in the form $P(x) = Q(x)D(x) + R(x)$.

Given $\frac{(x^3+2x^2-x+3)}{(x-2)} = (x^2 + 4x + 7) + \frac{17}{x-2}$,

write in the form $P(x) = (x-2)Q(x) + R(x)$.

$(x^3 + 2x^2 - x + 3) = (x^2 + 4x + 7)(x - 2) + 17$

a Given $\frac{x^3-x^2+x-1}{x+2} = x^2 - 3x + 7 - \frac{15}{x-2}$,
write in the form $P(x) = Q(x)D(x) + R(x)$

$(x^2 - 3x + 7)(x + 2) - 15$

e $(x^2 - 5x + 3) \div (x + 1) = (x - 6) + \frac{9}{x+1}$,
write in the form $P(x) = Q(x)D(x) + R(x)$

$(x - 6)(x + 1) + 9$

b Find $(2x^3 - x^2 + 3x - 1) \div (x + 3)$ and answer as $P(x) = (x - 2)Q(x) + R(x)$.

$(2x^2 - 7x + 24)(x + 3) - 73$

1 Complete the equation with the missing numbers.

a If 182 ÷ 3 = 60 remainder 2, then 182 = ……… × 60 + ………

$182 = 3 \times 60 + 2$

b If 2184 ÷ 5 = 436 remainder 4 , then 2184 = ………× 436 + ………

$2184 = 5 \times 436 + 4$

c If 617 ÷ 7 = 88 remainder 1, then 617 = 7 × ……… + ……….

$617 = 7 \times 88 + 1$

2 Perform each division and write the answer in the form $p = dq + r$

a $63 \div 5$

$63 = 5 \times 12 + 3$

b $125 \div 8$

$125 = 8 \times 15 + 5$

c $324 \div 11$

$324 = 11 \times 29 + 5$

3 Divide these polynomials and write in the form $P(x) = D(x)Q(x) + R(x)$.

a $(x^3 + x^2 - 2x + 3) \div (x - 1)$

$x^3 + x^2 - 2x + 3 = (x - 1)(x^2 + 2x) + 3$

b $(3x^3 - x^2 + x + 2) \div (x + 1)$

$3x^3 - x^2 + x + 2 = (x + 1)(3x^2 - 4x + 5) - 3$

c $(x^3 - x^2 + 2x + 1) \div (x + 1)$

$x^3 - x^2 + 2x + 1 = (x + 1)(x^2 - 2x + 4) - 3$

d $(x^4 - 3x^2 + x - 2) \div (x - 1)$

$x^4 - 3x^2 + x - 2 = (x - 1)(x^3 + x^2 - 2x - 1) - 3$

e $(x^3 - x + 1) \div (x + 2)$

$x^3 - x + 1 = (x^2 - 2x + 3)(x + 2) + (-5)$

f $(x^3 + x^2 - 3) \div (x - 1)$

$x^3 + x^2 - 3 = (x^2 + 2x + 2)(x - 1) + (-1)$

g $(x^4 - 2) \div (x + 3)$

$x^4 - 2 = (x^3 - 3x^2 + 9x - 27)(x + 3) + 79$

h $(x^4 - x^2) \div (x - 4)$

$x^4 - x^2 = (x^3 + 4x^2 + 15x + 60)(x - 4) + 240$

4 Divide these polynomials and write in the form $P(x) = D(x)Q(x) + R(x)$.

a $(3x^4 - 4x^3 + 4x - 8) \div (x^2 - 2)$

$$\begin{array}{r|l} & 3x^2 \\ \hline x^2 - 2 & 3x^4 - 4x^3 + 0x^2 + 4x - 8 \\ & 3x^4 \qquad\quad - 6x^2 \\ \hline \end{array}$$

$3x^4 - 4x^3 + 4x - 8 = (x^2 - 2)(3x^2 - 4x + 6) + (-4x + 4)$

b $(x^3 - 3x^2 + x + 4) \div (x^2 - x)$

$x^3 - 3x^2 + x + 4 = (x^2 - x)(x - 2) + (-x + 4)$

c $(x^3 + x^2 - 7x + 6) \div (x^2 + 3x - 1)$

$x^3 + x^2 - 7x + 6 = (x^2 + 3x - 1)(x - 2) + 4$

d $(x^4 - 3x^3 + x^2 - 7x + 3) \div (x^2 - 4x + 2)$

$x^4 - 3x^3 + x^2 - 7x + 3 = (x^2 - 4x + 2)(x2 + x + 3) + (3x - 3)$

e $(x^3 - 3x^2 + 5x - 4) \div (x^2 + 2)$

$x^3 - 3x^2 + 5x - 4 = (x^2 + 2)(x - 3) + (3x + 2)$

f $(2x^3 - 3) \div (2x - 4)$

$2x^3 - 3 = (2x - 4)(x^2 + 2x + 4) + 13$

5 If the divisor of a polynomial has degree 3, what are the possible degrees of the remainder?

Degree of the remainder must be less than the degree of the divisor.

∴ 0, 1, or 2

6 On division by $D(x)$, a polynomial has remainder $R(x)$ of degree 2.
What are the possible degrees of $D(x)$?

Degree of remainder must be less than degree of divisor.

∴ 3 or higher

7 $P(x) = x^3 - 3x - 2$

a Divide $P(x)$ by $x + 1$

b Complete the statement: $P(x) = (x + 1)(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$.

$(x^2 - x - 2)$

c Hence, by factorising the quadratic, express $P(x)$ as the product of three linear factors.

$(x + 1)^2(x - 2)$

8 Use long division to show that $P(x) = x^3 + 2x^2 - 11x - 12$ can be divided by $x - 3$ with no remainder and hence express $P(x)$ as the product of three linear factors.

$x^3 + 2x^2 - 11x - 12 = (x + 4)(x + 1)(x - 3)$

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$P(x)$ is a polynomial of degree 5. When $P(x)$ is divided by a polynomial $Q(x)$, the remainder is $2x+5$. Which statement is true about the degree of Q?

A. The degree must be 1.

B. The degree could be 1.

C. The degree must be 2.

D. The degree could be 2.

The remainder has degree 1.

The degree of the remainder is smaller than the degree of the divisor.

D.

10

a Use long division to show that $P(x)=2x^4+3x^3-12x^2-7x+6$ is divisible by x^2-x-2 and express $P(x)$ as a product of four linear factors.

$(x-2)(x+1)(2x-1)(x+3)$

11

a Find the quotient and remainder when $x^4-2x^3+x^2-5x+7$ is divided by x^2+x-1.

quotient: x^2-3x+5, remainder: $12-13x$

b Hence find a and b such that $x^4-2x^3+x^2+ax+b$ is exactly divisible by x^2+x-1.

Consider the long division, the remainder comes from $(ax-3x-5x)+(b+5)$

We want remainder to be zero, so $ax-3x-5x=0$

$b+5=0$

$\therefore a=8, b=-5$

12 If $x^4 - 2x^3 - 20x^2 + mx + n$ is exactly divisible by $x^2 - 5x + 2$, find m and n.

From long division, the remainder is $(m - 41)x + (n + 14)$

$\therefore m = 41, n = -14$