Student Name

Mathematics Stage 5 Core

NON-LINEAR RELATIONSHIPS B

Book 3 Distinguish between linear, quadratic, and exponential relationships by examining their graphical representations

Version: 260207

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CONTENTS

OVERVIEW

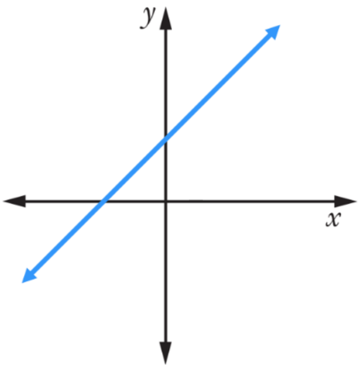
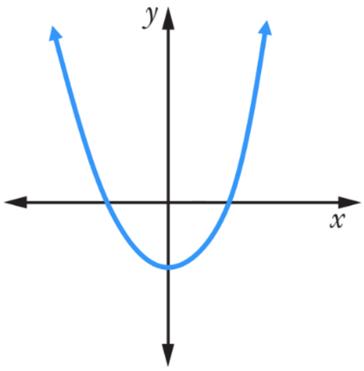
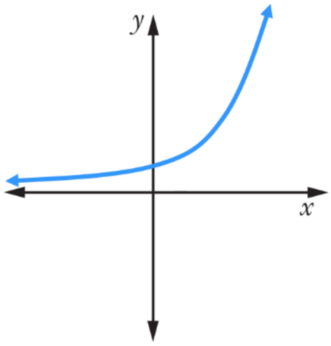
SYLLABUS CONTENT

MA5-NLI-C-02 identifies and compares features of parabolas and exponential curves in various contexts

Distinguish between linear, quadratic and exponential relationships by examining their graphical representations

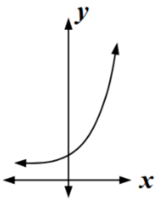
- Associate graphs of straight lines, parabolas and exponential curves with the appropriate equations
- Recognise non-linear relationships in real-life contexts and solve related problems
- Use graphing applications to solve a pair of simultaneous equations where one equation is non-linear and interpret the solution

COMPARING RELATIONSHIPS

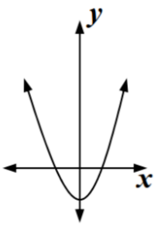
Linear Relationship	Quadratic Relationship	Exponential Relationship
$y = mx + c$	$y = ax^2 + bx + c$	$y = a^x$
		

1 Decide whether the following are straight lines, parabolas, or exponential curves.

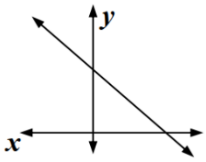
a



b



c



d $y = 2x^2$

e $y = 5^x$

f $y = 4x + 2$

g $x = -3$

h $y = 3^x$

i $y = x$

j $y = x^2 + 3x - 5$

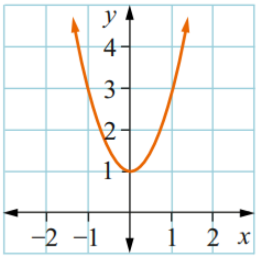
k $y = 4 - 2x$

l $y = -2^x$

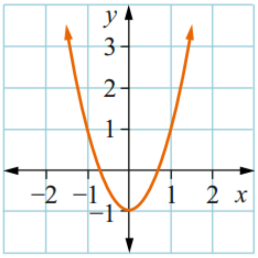
Example Identify the graph from an equation.

Determine which of the following could be the graph of $y = 2x^2 - 1$.

A



B



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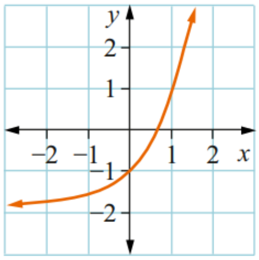
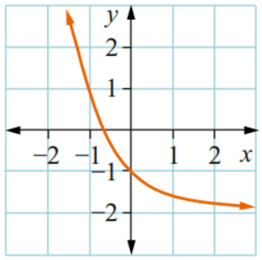


D



Why can you eliminate C and D?

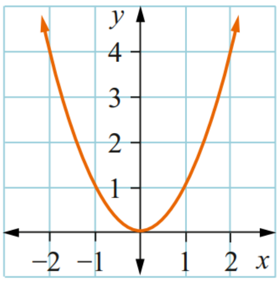
………

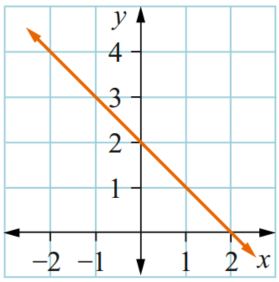
How do you know whether it is A or B?

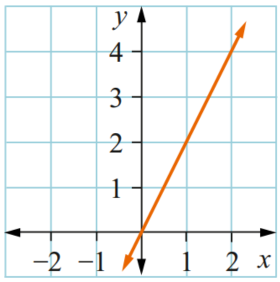
………

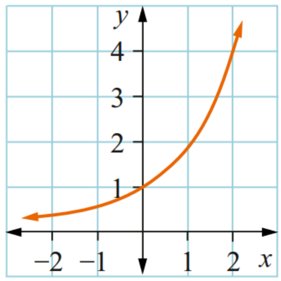
Match the equations with the graphs below:

a $y = 2 - x$	**b** $y = 2x$	**c** $y = x^2$	**d** $y = 2^x$







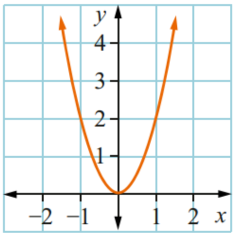


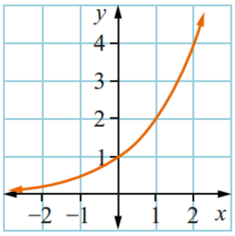
Key ideas

1. Equations, when graphed, have a specific ………………………
2. Conversely, we can determine the …………………from the shape of the graph.

1 Circle which of the following could be the graph of:

a $y = 2^x$

A


B


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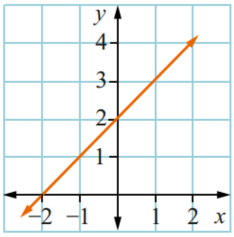
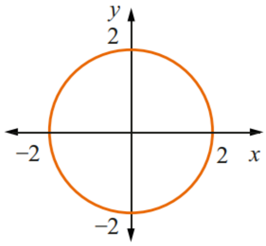
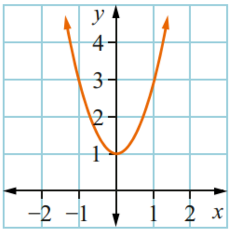
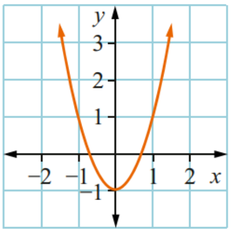
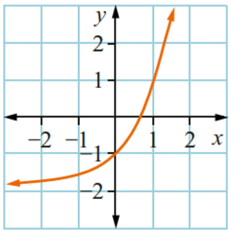


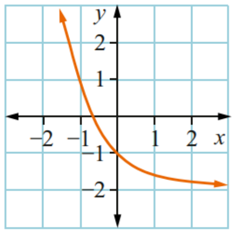
D


b $y = 2x^2 + 1$

A


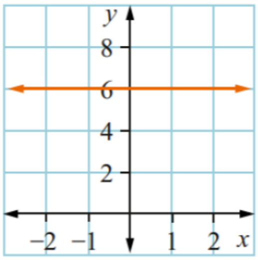
B


C


D


2 Match the equations with the graphs.

$y = 2x^2$ $y = 6$ $y = x + 1$ $y = x$ $y = 2^{-x}$

a


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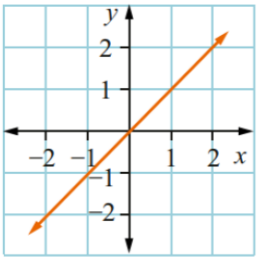

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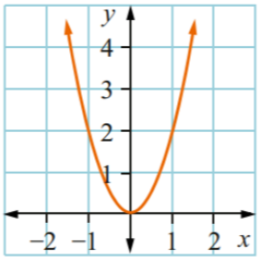
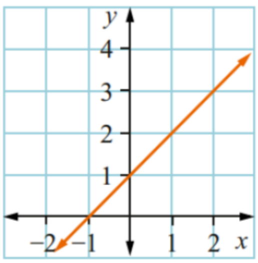
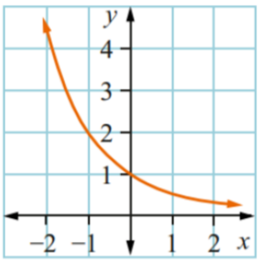


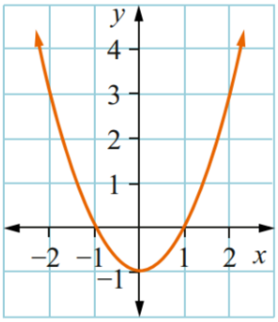
d


e


3 Match the equations with the graphs.

$y = 3^x$ $\quad$ $y = x^2 + 1$ $\quad$ $y = x + 2$ $\quad$ $y = x^2 - 1$ $\quad$ $y = 1 - x^2$

a



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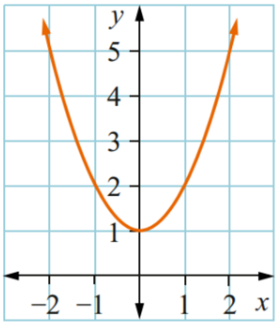

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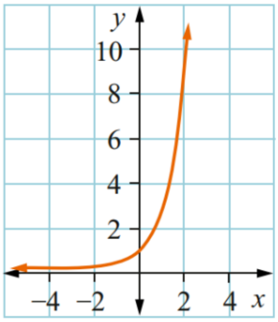

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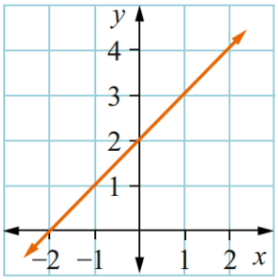

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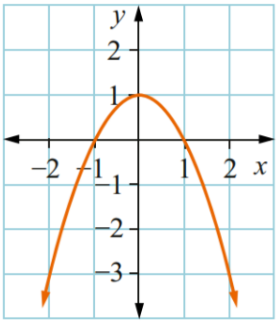
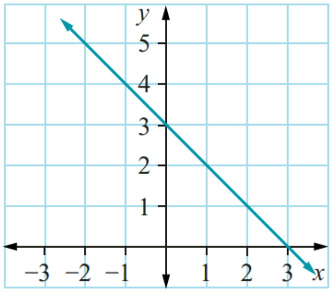


4 Match the equations with the graphs.

$y = 3 - x$ $\quad$ $y = -2x^2$ $\quad$ $y = 4x^2 - 1$ $\quad$ $y = 5^{-x}$ $\quad$ $y = 3^x + 1$

a



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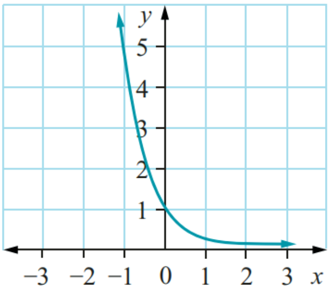

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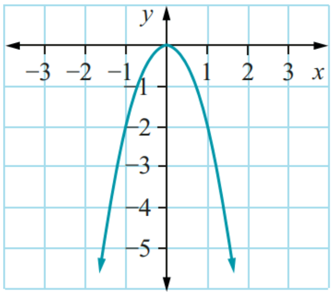

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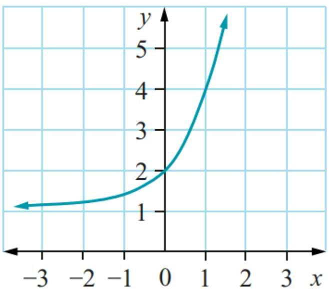

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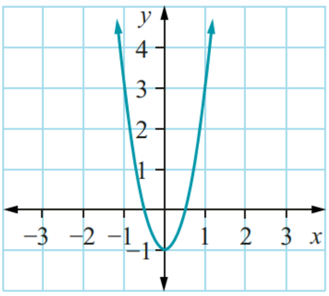
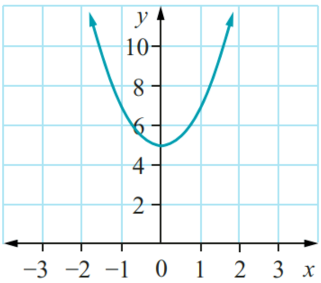


5 Match the equations with the graphs.

$y = 2x - 4$ $\quad$ $y = -3x^2$ $\quad$ $y = 2x^2 + 5$ $\quad$ $y = -5^{-x}$ $\quad$ $y = 5^x - 1$

a



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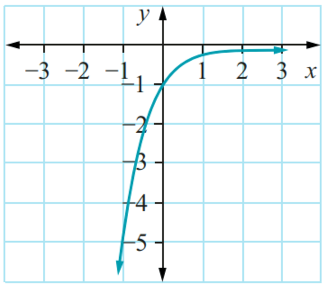
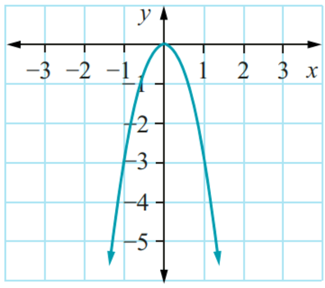


c



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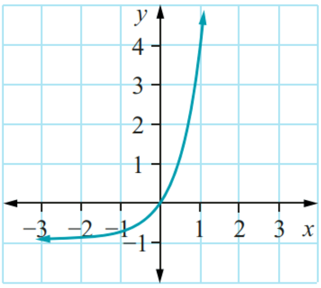

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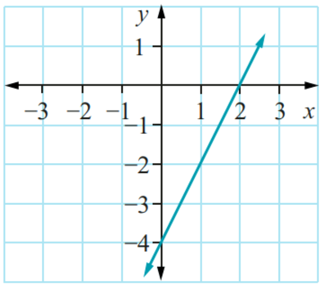
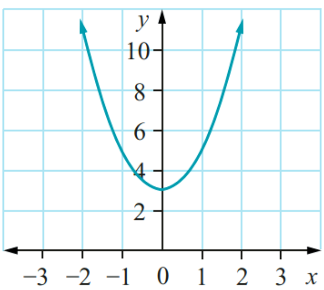


6 Match the equations with the graphs.

$x + 2y - 4 = 0$ $\quad$ $y = 2x^2 + 3$ $\quad$ $y = -2x^2 - 3$ $\quad$ $y = -2^{-x} + 1$ $\quad$ $y = -2^x - 1$

a



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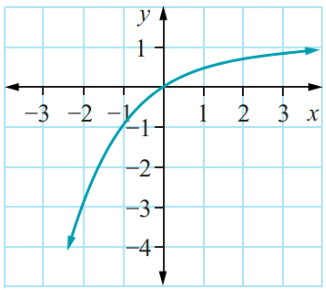

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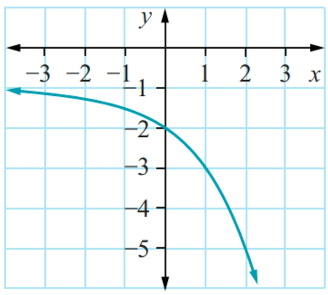

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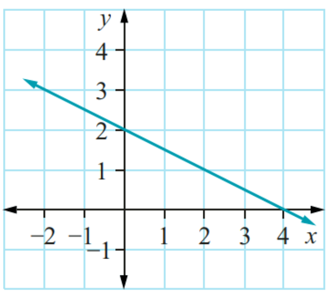

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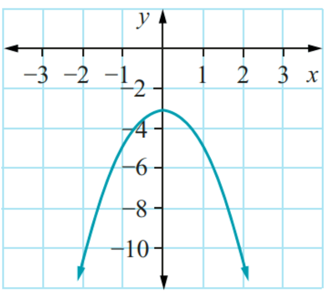
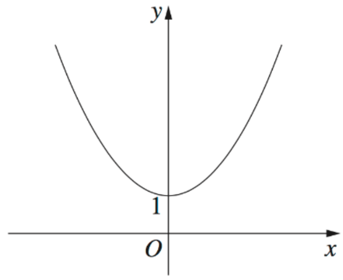


7 2014 HSC Standard 2 Band 3

Which of the following equations does the graph best represent?



A. $y = \frac{3}{x} + 1$

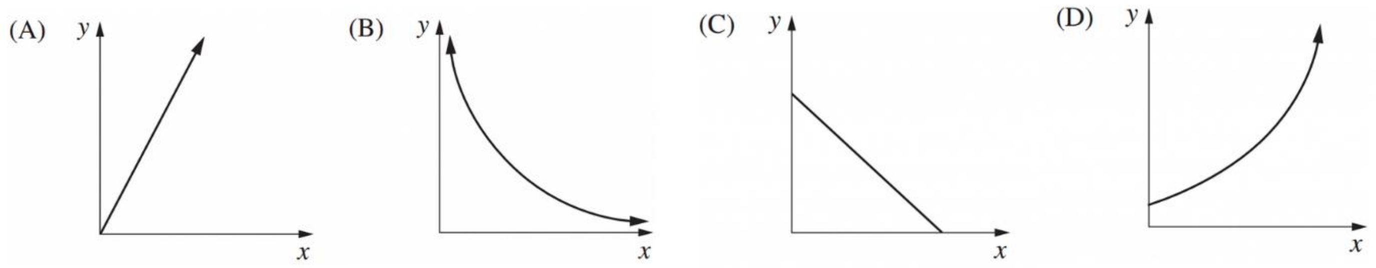
B. $y = 3^x + 1$

C. $y = 3x^2 + 1$

D. $y = 3x^3 + 1$

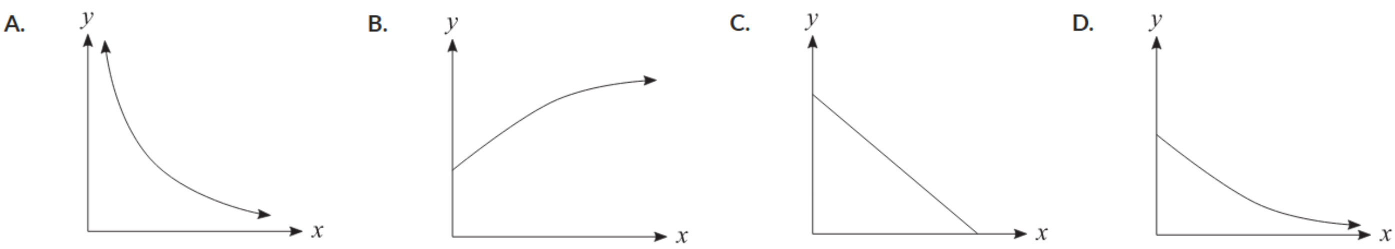
8 2008 HSC Standard 2 Band 4

Which graph best represents $y = 3^x$?



9 HSC Sample Question Band 5

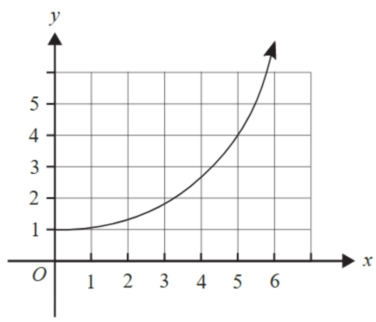
Which graph best represents $y = 5^{-x}$?



10 HSC Sample Question Band 5

The graph shows $y = a^x$.

What is the approximate value of $2(a^5)$?

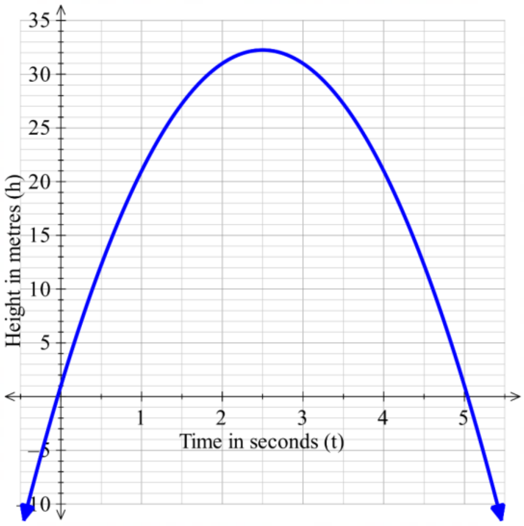


QUADRATIC MODELLING

- A **quadratic model** uses a quadratic equation to show a real situation.
- It finds **maximum** or **minimum** values using the **turning point**.

Example Interpret a quadratic model.

A girl throws a ball in the air, the ball travels upwards, then falls back down.
The height of the ball is modelled using the equation $h = -5t^2 + 25t + 1$.



Find the maximum height of the ball. 32 m

When is the ball at the maximum height? 2.5 s

At which two times is the ball 31 m high? 2 s & 3 s

At what time is the height zero? 0 s & 5 s

Calculate the height when $t = 3.5$

$h = -5(3.5)^2 + 25(3.5) + 1$

$= 25.72$ m

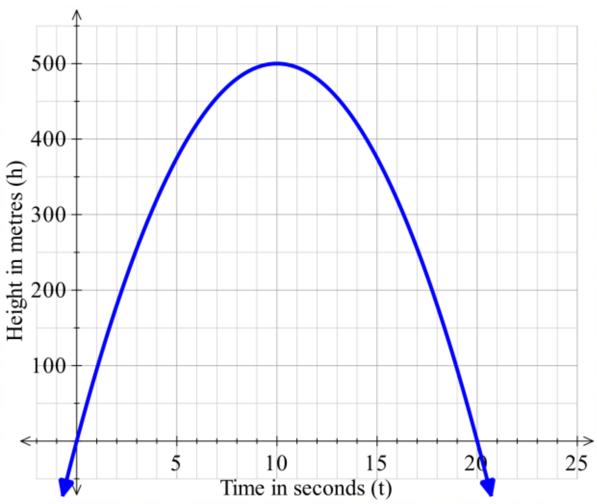
What are the limitations of this model?

$0 \le t \le 5$, as time cannot be negative, and height cannot be negative (unless the ball goes underground, which doesn't make sense in this scenario).

The turning point is always halfway between the x-intercepts. Explain why.

The turning point is halfway between x-intercepts because a parabola is …………………

A rocket is launched into the air. The height of the rocket is $h = -5t^2 + 100t$, where h is height in metres and t is time in seconds.



What is the maximum height and when does it occur?

500 metres at 10 seconds

At which two times is the ball 375 m above the ground?

5 and 15 seconds

At what times is the height zero?

0 and 20 seconds

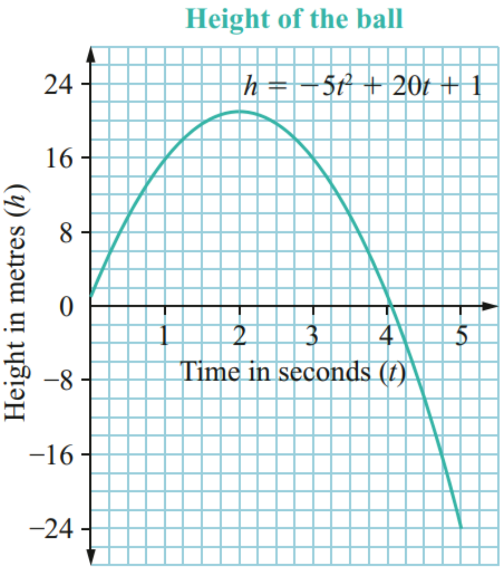
What are the limitations of the model?

$0 \le t \le 20$

Key ideas

1. Quadratic models are used to find the ………………… or …………………… value.
2. The symmetric nature of a parabola allows us to find the vertex or the …… - ………………

1 A ball is thrown into the air. The height of the ball, h, in metres, can be modelled using the quadratic equation $h = -5t^2 + 20t + 1$, where t is time in seconds.



a Find the maximum height of the ball.

21 m

b When is the ball at its maximum height?

2 s

c At which two times is the ball 16 m above the ground.

1 s and 3 s

d At what time is the height zero?

0 s and 4 s

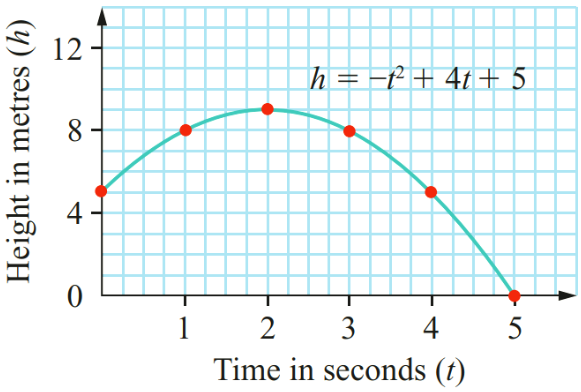
e What is the value of h when $t = 5$?

−24 m

f What limitations are there on this model?

$0 \leq t < 4$; the height can't be negative; ball can't go underground.

2 An object is thrown into the air. The height of the object can be modelled using the equation $h = -t^2 + 4t + 5$, where h is the height in metres and t is the time in seconds.



a Find the maximum height.
When does this occur?

9 m at 2 s

b What was the initial height of the object?

5 m

c When does the object hit the ground?

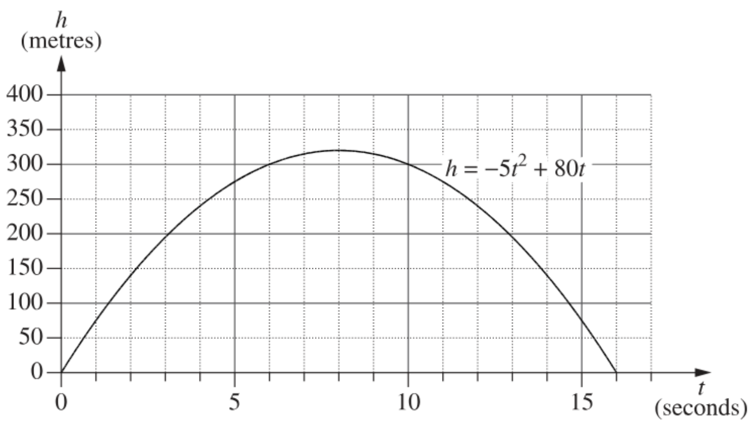
5 s

d Why does the graph not show the whole parabola?

Time can't be negative and height can't be negative.

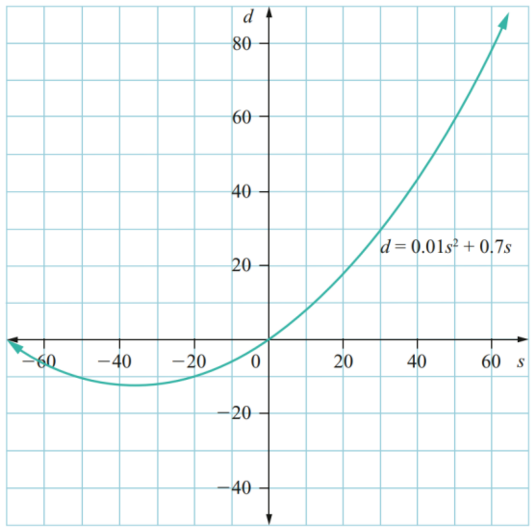
3 **2022 HSC Standard 2 Band 3**

An object is projected vertically into the air. For how long is the object at a height of 300 m or more above the ground?



4 s

4 Nadia is investigating stopping distances for a car travelling at different speeds.
To model this she uses the equation $d = 0.01s^2 + 0.7s$, where d is the stopping distance in metres and s is the speed of the car in km/h. The graph of this equation is drawn on the right.



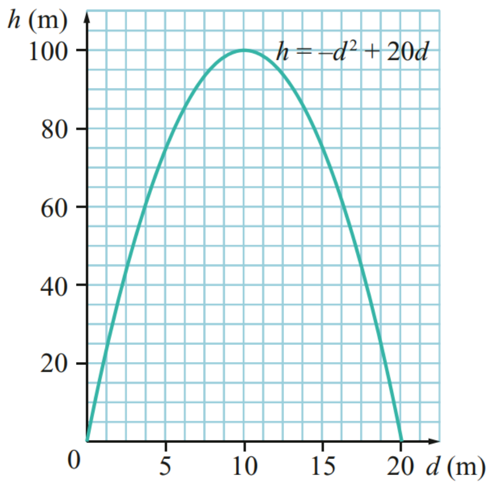
a Nadia knows that only part of this curve applies to her model for stopping distances.
Highlight the part of this curve that applies for stopping distances.

$s \geq 0$

b What is the difference between the stopping distances in a school zone when travelling at a speed of 40 km/h and when travelling at a speed of 60 km/h?

≈ 34 m

5 An amateur golfer hits a ball up into the air.
The path of the ball follows the relationship $h = -d^2 + 20d$, where h is the height of the ball for a horizontal distance d from where the ball was hit. Both h and d are in metres.



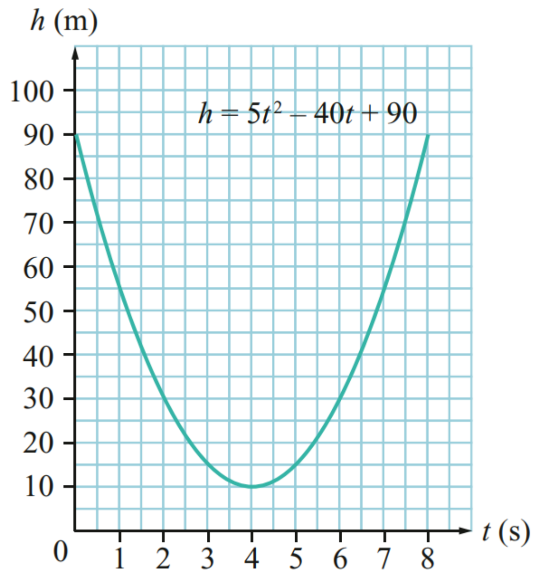
a What was the maximum height of the golf ball?

100 m

b How far from the golfer did the ball land?

20 m

6 The height above the ground for a bungee jumper is modelled by $h = 5t^2 - 40t + 90$ where h is the height in metres after time t seconds.



a Use the graph to find the height of the person above the ground at the start of the jump.

90 m

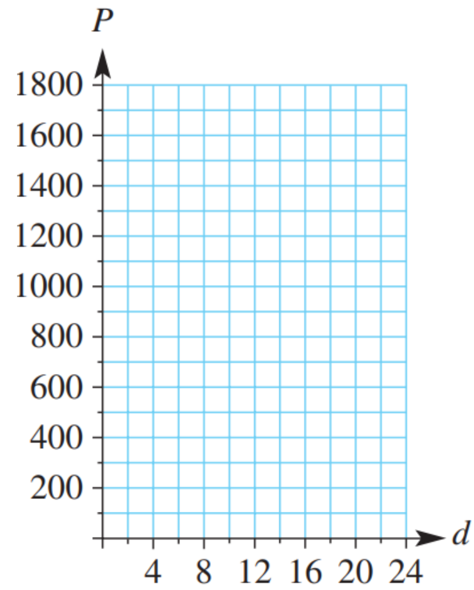
b Use the graph to find the lowest height above the ground that the person falls to in the first downwards movement of the jump.

10 m

7 The price ($\$P$) of rope depends on the diameter (d), in metres, of the rope when it is rolled into a circle. The quadratic equation $P = 3d^2$ is used to model this situation.

a Fill in the table and sketch the curve.

d	0	2	4	6	8	10	12	14	16	18	20	22	24
P													



b What is the price of the rope when the diameter of the rope is 12 metres?

$432

c What is the price of the rope when the diameter of the rope is 23 metres?

$1587

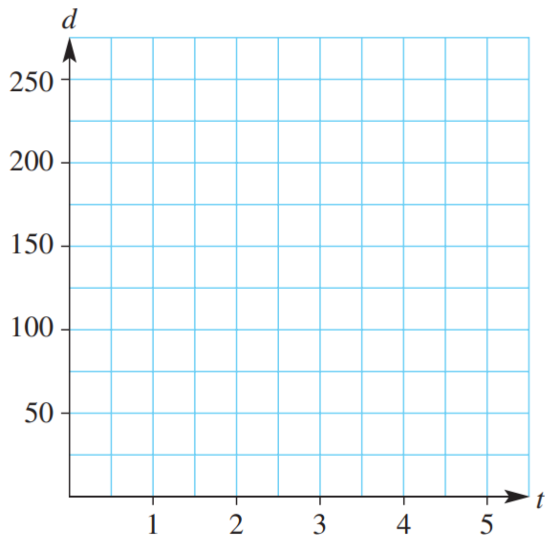
d What is the difference between the price of the rope when the diameter of the rope is 5 metres compared to a diameter of 15 metres?

$600

8 A stone falls from rest down a mine shaft. The distance it falls, d metres, at time t seconds is given by the quadratic equation $d = 4.9t^2$

a Fill in the table and sketch the curve.

t	0	1	2	3	4	5
d						



b What is the distance travelled by the stone after 1 second?

4.9 m

c What is the distance travelled by the stone after 5 seconds?

122.5 m

d What is the distance travelled by the stone after 2.5 seconds?

30.625 m

e Use the graph to estimate how long it takes for the stone to travel 100 metres.

≈ 4.5 s

f Use the equation to calculate how long it takes for the stone to travel 200 metres.

≈ 6.4 s

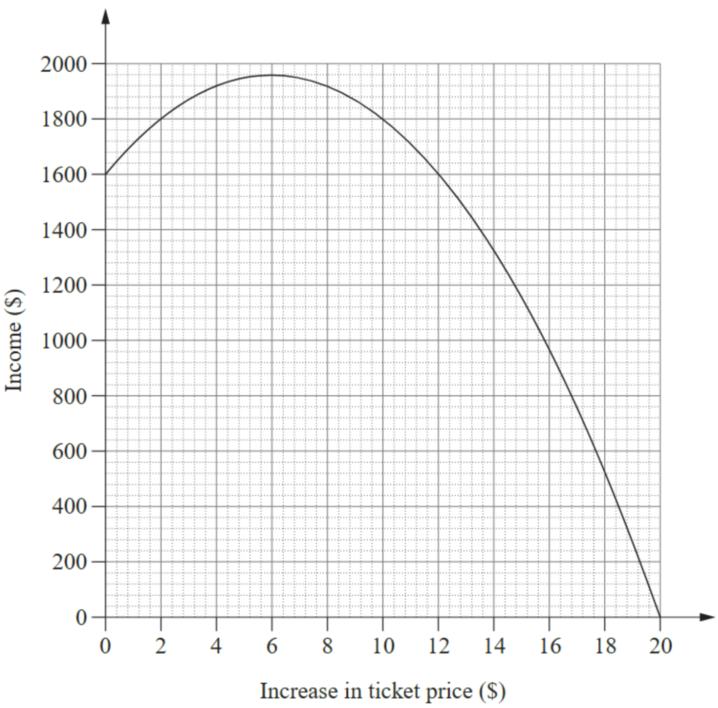
9 2017 HSC Standard 2 Band 5

A movie theatre has 200 seats. Each ticket currently costs \$8.

The theatre owners are currently selling all 200 tickets for each session.
They decide to increase the price of tickets to see if they can increase the income earned from each movie session.

It is assumed that for each one dollar increase in ticket price, there will be 10 fewer tickets sold.

A graph showing the relationship between an increase in ticket price and the income is shown below.



What ticket price should be charged to maximise the income from a movie session?

\$8 + \$6 = \$14

What is the number of tickets sold when the income is maximised?

$1960 \div 14 = 140$

The cost to the theatre owners of running each session is \$500 plus \$2 per ticket sold.

Calculate the profit earned by the theatre owners when the income earned from a session is maximised.

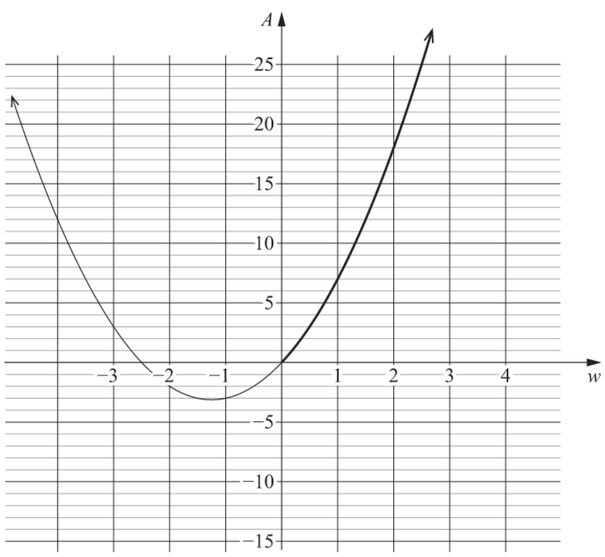
Cost: $500 + 2 \times 140 = \$780$

Profit: $1960 - 780 = \$1180$

10 2019 HSC Standard 2 Band 5

A rectangle has width w centimetres. The area of the rectangle A in square centimetres is $A = 2w^2 + 5w$. The graph is shown.

Explain why, in this context, the model only makes sense for the bold section of the graph.



Width of a rectangle can't be negative.

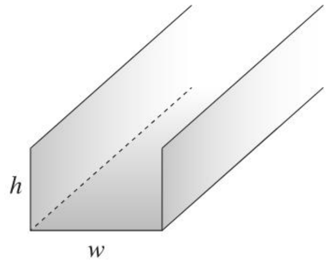
The area of the rectangle is 18 cm².

Calculate the perimeter of the rectangle.

22 cm

11 2024 HSC Standard 2 Band 5

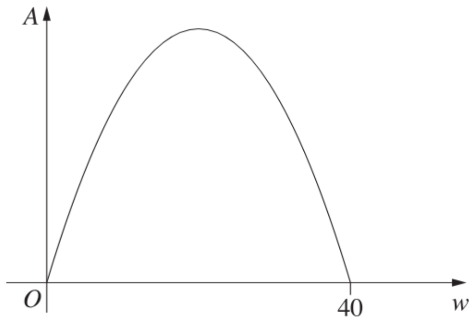
A sheet of metal is folded to make a gutter, as shown. The cross-section of the gutter is a rectangle of width w cm and height h cm.



The area, A cm², of the cross-section can be modelled by the quadratic formula

$$A = -0.5w^2 + 20w$$

A graph of this model is shown.



Find the width and height of the rectangle which will give the greatest possible area of the cross-section.

Max area occurs at $w = 20$ cm

Substitute $w = 20$ into equation to get $A = 200$ cm^2

Since $A_{\text{rect}} = wh$,

$h = A \div w = 10$ cm

$\therefore w = 20 \text{ cm}, h = 10 \text{ cm}$

12 2023 HSC Standard 2 Band 5

The braking distance of a car, d, in metres, is directly proportional to the square of its speed s in km/h, and can be represented by the equation

$$d = ks^2$$

Where k is the constant of variation.

The braking distance of a car travelling at 50 km/h is 20 m.

What is the braking distance when the speed of the car is 90 km/h?

64.8 m

13 HSC Sample Question Band 4

Moses finds that for a Froghead eel, its mass is directly proportional to the square of its length.

An eel of this species has a length of 72 cm and a mass of 8250 grams.

What is the expected length of a Froghead eel with a mass of 10.2 kg?
Give your answer to one decimal place.

80.1 cm

14 2009 HSC Standard 2 Band 5

The height above the ground, in metres, of a person's eyes varies directly with the square of the distance, in kilometres, that the person can see to the horizon.
A person whose eyes are 1.6 m above the ground can see 4.5 km out to sea.

How high above the ground, in metres, would a person's eyes need to be to see an island that is 15 km out to sea? Give your answer correct to one decimal place.

17.8 m

EXPONENTIAL MODELLING

- An **exponential model** uses an exponential equation to show a real situation.
- Examples of exponential models include population growth, compound interest, declining-balance depreciation and radioactive decay.

Exponential Growth	Exponential Decay
$y = a(1 + r)^x$ a: Initial value r: Increase per period, as a decimal.	$y = a(1 - r)^x$ a: Initial value r: Decrease per period, as a decimal.

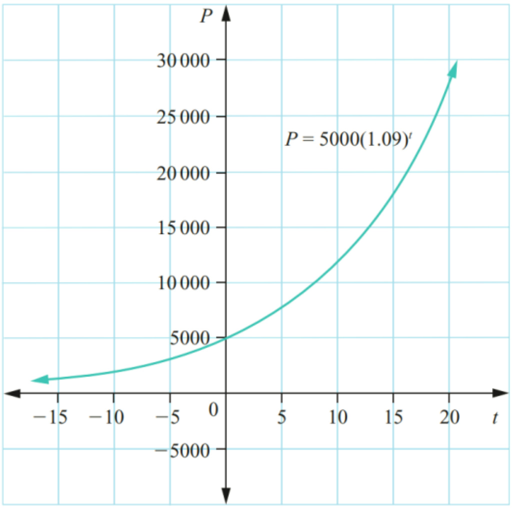
Example Identify exponential growth or decay.

In each model, identify initial value, growth or decay, and increase/decrease per period.

a $y = 20(1 + 0.1)^x$ Initial value: 20 Growth Decay Change per period: +10%	**b** $y = 40(1 + 0.2)^x$ Initial value: Growth Decay Change per period:	**c** $y = 40(1 - 0.2)^x$ Initial value: Growth Decay Change per period:
d $y = 35(1 + 0.5)^x$ Initial value: Growth Decay Change per period:	**e** $y = 1400(1 - 0.32)^x$ Initial value: Growth Decay Change per period:	**f** $y = 7600(1 - 0.44)^x$ Initial value: Growth Decay Change per period:
g $y = 3500(1.12)^x$ Initial value: Growth Decay Change per period:	**h** $y = 2300(1.72)^x$ Initial value: Growth Decay Change per period:	**i** $y = 180(0.8)^x$ Initial value: Growth Decay Change per period:
j $y = 800(0.6)^x$ Initial value: Growth Decay Change per period:	**k** $y = 1240(0.65)^x$ Initial value: Growth Decay Change per period:	**l** $y = 5000(1.175)^x$ Initial value: Growth Decay Change per period:

Example Interpret an exponential model.

The population of an island is modelled using the equation $P = 5000(1.09)^t$ where P is the population and t is the time in years.



What are the limitations of the model?

$t \geq 0$; time cannot be negative.

Growth is also unlikely to continue exponentially for long periods of time.

Does the graph show exponential growth or decay?

Exponential growth, as graph is increasing.

What was the initial population?

at $t = 0, P = 5000$

Use the graph to estimate population after 10 years.

at $t = 10, P \approx 12000$

Use the equation to calculate the population after 10 years.

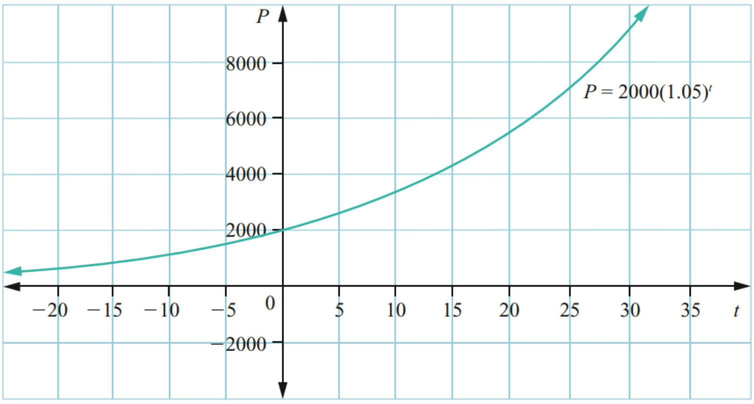
$P = 5000(1.09)^{10} = 11836.82$

Use the graph to estimate when the population reaches 20 000. Around 16 years.

The population is increasing by 9% each year. What part of the equation tells you this?

The .... number is 1.09, which is what you .... by when increasing by 9%.

The population of a town is $P = 2000(1.05)^t$ where P is the population and t is in years.



What was the initial population?

2000

Estimate the population after 25 years.

Around 7000

Calculate the population after 25 years using the equation.

6773

How long does it take the population to double?

Around 14 years

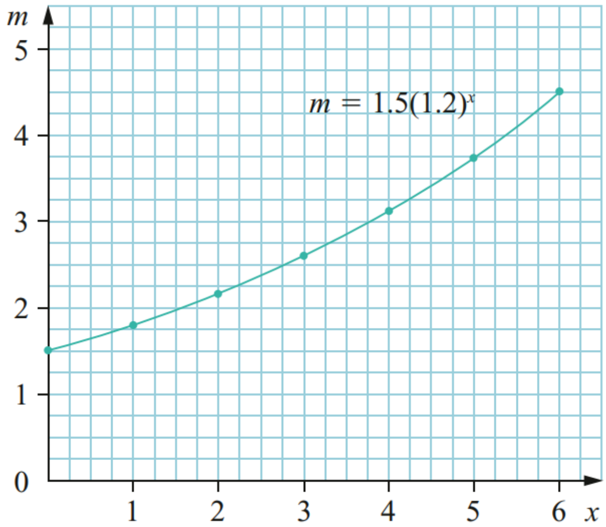
What is the percentage growth each year?

5%

Key ideas

1. Exponential models are used to analyse repeated .... change.

1 The equation $m = 1.5(1.2)^x$ can be used to model the mass of a baby orangutan up to 6 months of age, where m is the mass in kilograms and x is the age in months.



a Estimate the mass of the orangutan at age 3.5 months.

≈ 2.8 kg

b At about what age would the orangutan weigh 3.4 kg?

≈ 4.5 months

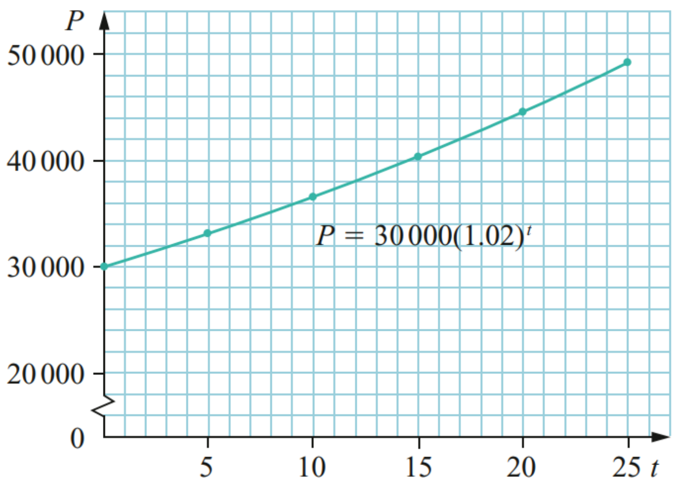
c Each month, the mass increases by a factor of 1.2. What is the monthly growth rate of the baby orangutan's mass? Write your answer as a percentage.

20%

d Why might this model only be useful up to age 6 months?

Growth rate of an orangutan might speed up or slow down after 6 months.

2 The population of an island is modelled using the equation $P = 30\,000(1.02)^t$, where P is the population and t is the time in years.



a What does the intercept of the curve with the vertical axis represent?

Initial population

b Use the graph to estimate the population after 18 years.

≈ 43 000

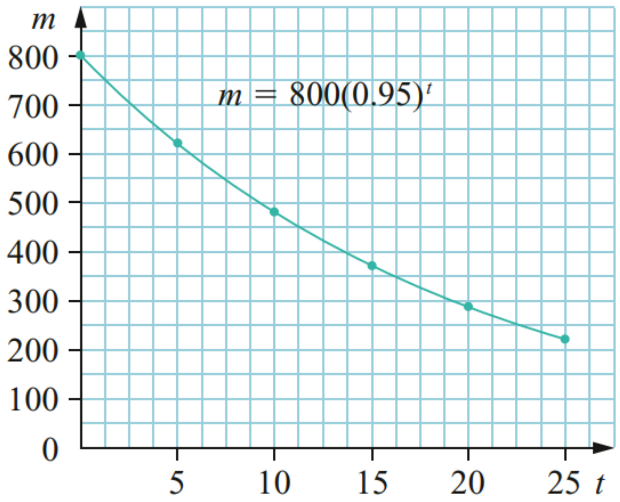
c Estimate the time taken for the population to reach 39 000.

≈ 14 months

d What is the percentage increase in population each year?
Explain how you can work this out from looking at the equation $P = 30\,000(1.02)^t$

1.02 means an increase of 2%

3 Radioactive decay is a physical phenomenon that can be modelled using an exponential relationship. A piece of radioactive material decays according to the equation $m = 800(0.95)^t$, where m is the mass of radioactive substance in grams and t is the number of years.



a Explain how we can see that this model represents exponential decay.

The graph is decreasing; the equation has a base that is less than 1: 0.95.

b What was the initial mass of radioactive material?

800 g

c Use the graph to estimate the mass of radioactive substance after 22.5 years.

≈ 250 g

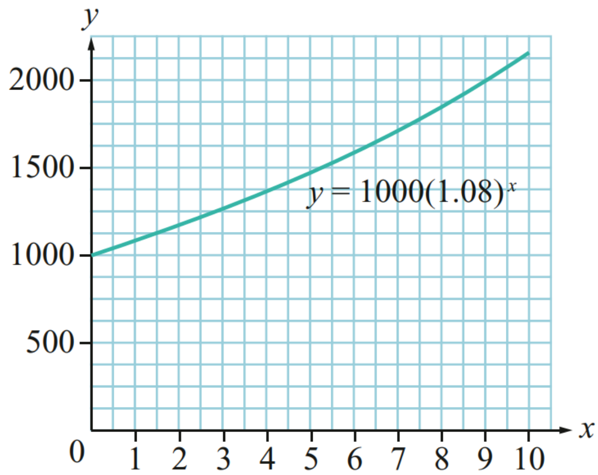
d Estimate the time taken for the mass of radioactive substance to reach 600 g.

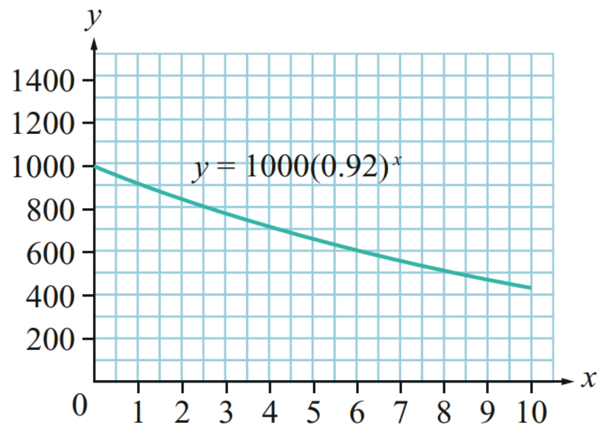
≈ 6 years

e What is the percentage rate of decay each year?
Explain how you can work this out from looking at the equation $m = 800(0.95)^t$

0.95 means 5% decrease.

4 The two graphs $y = 1000(1.08)^x$ and $y = 1000(0.92)^x$ are shown.





a Which of these graphs would be used to model depreciation? How do you know?

Second graph, shows exponential decay.

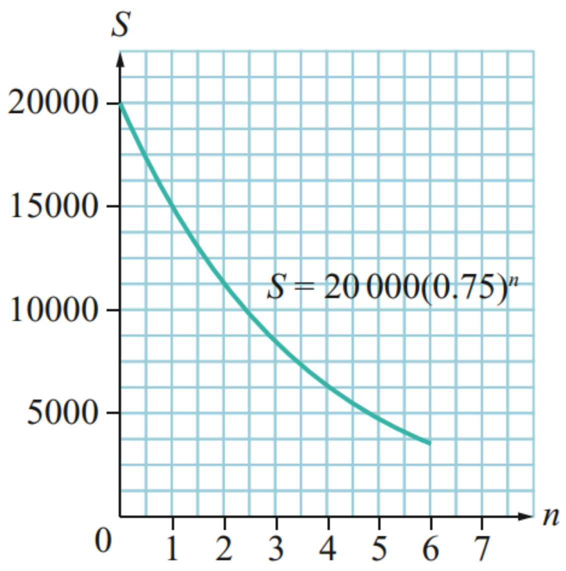
b Which of these graphs would be used to model inflation? How do you know?

First graph, shows exponential growth.

c What would be the difference in the value of two items after 8 years, if both were purchased for \$1000 and one appreciated at 8% p.a. and the other depreciated at 8% p.a.?

≈ \$1300

5 A new motorcycle is purchased for $20 000 and depreciates in value at a rate of 25% per year. The salvage value of the motorcycle can be modelled using the equation $S = 20\ 000(0.75)^n$



a Does this graph represent exponential growth or decay?

Decay.

b What does the intercept of the curve with the vertical axis represent?

Initial value of the motorcycle.

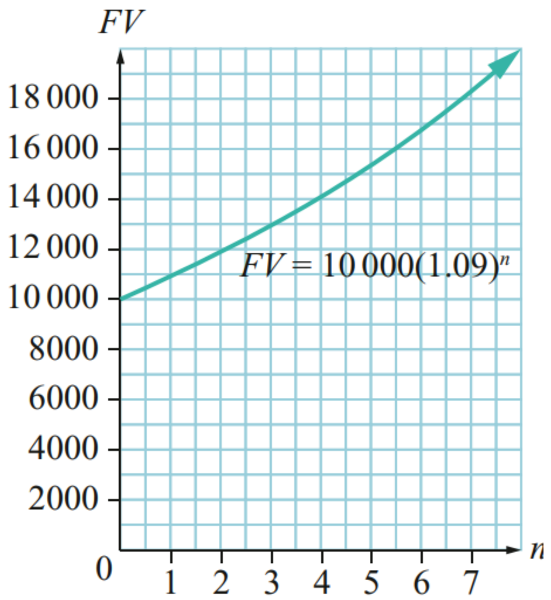
c Use the graph to find the salvage value of the motorcycle after 2 years.

≈ $11 250

d From the graph, estimate the number of years it takes for the salvage value of the motorcycle to be less than $6000.

≈ 4 years

6 The compound interest formula is $FV = PV(1 + r)^n$. The graph of a compound interest investment is shown.



a Looking at the equation of the graph, what was the present value of the investment?

$10 000

b Looking at the equation of the graph, what is the percentage interest rate per period?

9% increase

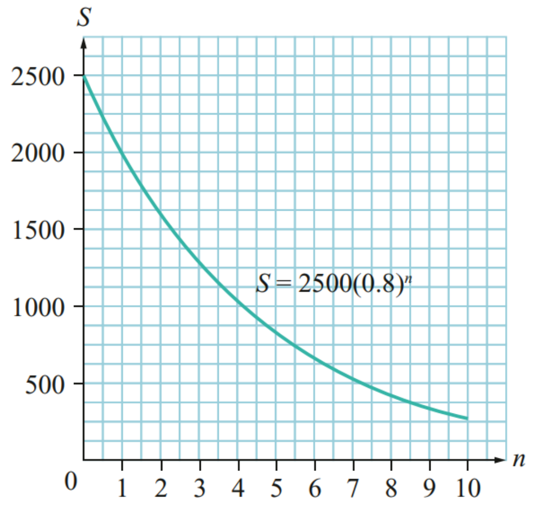
c Estimate the time it takes for the future value to grow to 14 000.

≈ 4 years

d Estimate the future value after 7 years using the graph, then check using the equation.

$18 280.39

7 The formula for declining balance depreciation is $S = V_0(1-r)^n$. The graph of a declining balance depreciation is shown.



a Looking at the equation of the graph, what was the initial value of the investment?

$2500

b Looking at the equation of the graph, what is the percentage depreciation rate per period?

20%

c Estimate the time it takes for the salvage value to reach $500.

≈ 7 years

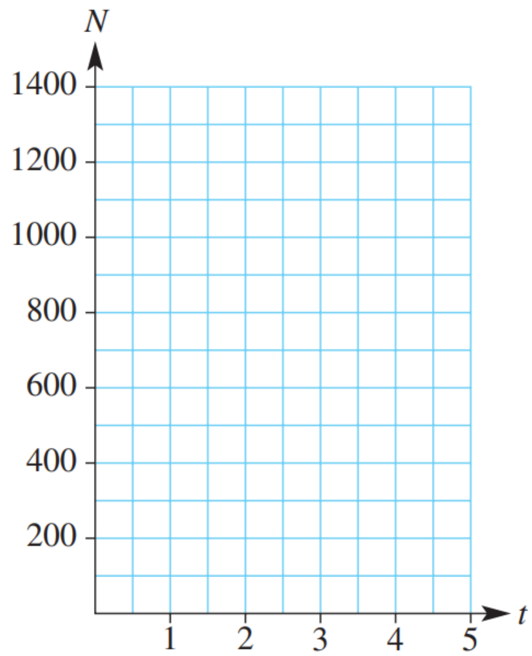
d Estimate the salvage value after 3 years using the graph and check using the equation.

$1280

8 The exponential function, $N = 6^t$ is used to model the growth in the number of insects (N) after t days.

a Fill in the table and sketch the curve.

t	0	1	2	3	4
N					



b What was the initial number of insects?

1

c How many insects were present after 3 days?

216

d How many extra insects will be present after 4 days compared to 2 days?

1260

e Estimate how many days it took for the number of insects to exceed 1000. Answer to the nearest 1 d.p.

≈ 3.9 days

9 2021 HSC Standard 2 Band 5

A population P is to be modelled using the function $P = 2000(1.2)^t$ where t is the time in years.

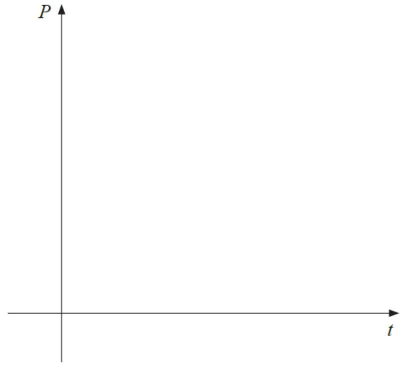
What is the initial population?

2000

Find the population after 5 years. Answer to the nearest whole.

4977

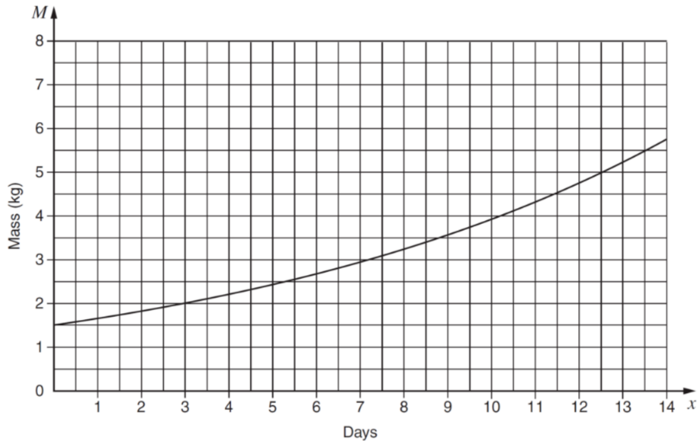
On the axes, sketch the population against time, showing the points at $t = 0$ and $t = 5$



Exponential growth curve with points (0, 2000) and (5, 4977)

10 2016 HSC Standard 2 Band 5

The mass of a baby pig at age x days is given by $M = A(1.1)^x$ where A is a constant.
The graph is shown.



What is the value of A?

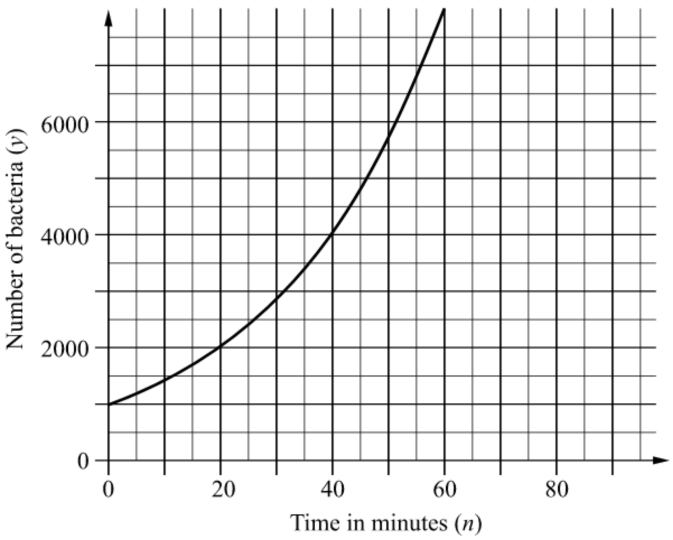
1.5

What is the daily growth rate of the pig's mass? Write your answer as a percentage.

10%

11 2020 HSC Standard 2 Band 6

The graph shows the number of bacteria y at time t minutes. Initially, when $n = 0$, the number of bacteria is 1000.



Find the number of bacteria at 40 minutes.

4000

The number of bacteria can be modelled by $A \times b^n$ where A and b are constants.

Use guess and check to find, to two decimal places, an upper and lower estimate for the value of b. The upper and lower estimates must differ by 0.01.

We want to find a value of b such that $1000 \times b^{40} \approx 4000$

$1.03 < b < 1.04$