Student Name

Mathematics Stage 5 Paths

# NON-LINEAR RELATIONSHIPS C

**Book 5** Distinguish between different types of graphs by examining their algebraic and graphical representations and solve problems

Version: 260207

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## CONTENTS

# OVERVIEW

## SYLLABUS CONTENT

**MA5-NLI-P-01** interprets and compares non-linear relationships and their transformations, both algebraically and graphically

**Distinguish between different types of graphs by examining their algebraic and graphical representations and solve problems**

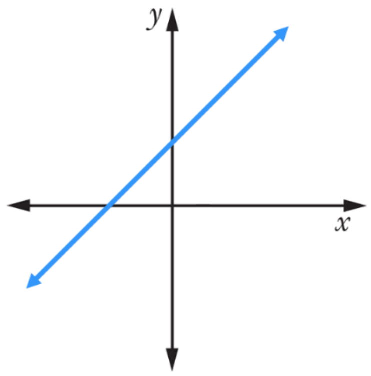
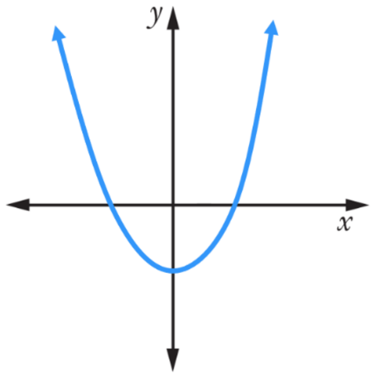
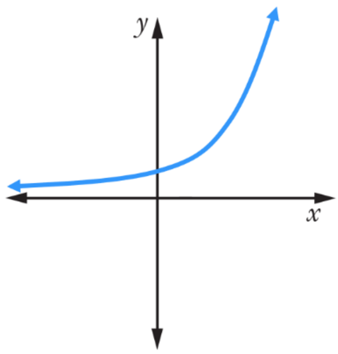
- Identify and describe features of different types of graphs based on their equations
- Identify a possible equation from a graph and verify using graphing applications
- Find points where a line intersects with a parabola, hyperbola or circle, both graphically and algebraically

## REVIEW OF PRIOR KNOWLEDGE

**1** Rearrange the following into the form $ax^2 + bx + c = 0$ , where $a > 0$.

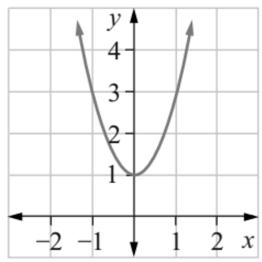
| | | |
|---|---|---|
| **a** $x^2 + 5x = 2x - 6$ | **b** $x^2 - 3x + 4 = 2x + 1$ | **c** $-x^2 + 2 = 2x + 5$ |

# EQUATIONS AND THEIR GRAPH

| Linear Relationship | Quadratic Relationship | Exponential Relationship |
|---|---|---|
| $y = mx + c$ | $y = ax^2 + bx + c$ | $y = a^x$ |
|  |  |  |

**Example** Identify the graph from an equation.

Determine which of the following could be the graph of $y = 2x^2 - 1$

A
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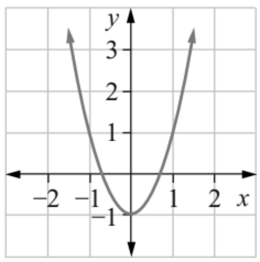


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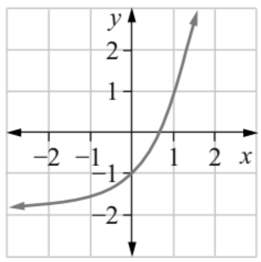
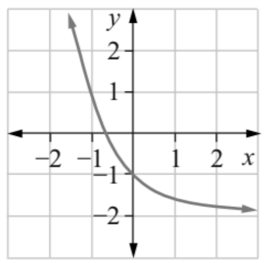



D



Why can you eliminate C and D?

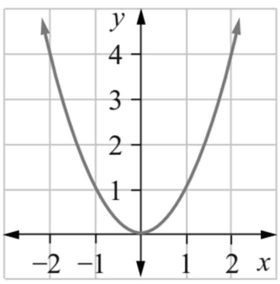
*The equation has $x$ to the power of ………, so it must be a ………………, so not C or D.*

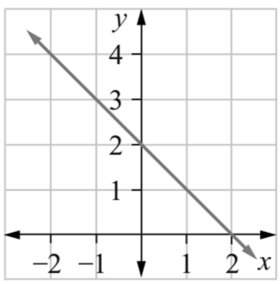
How do you know whether it is A or B?

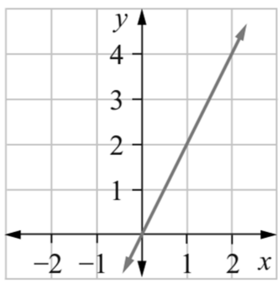
*By substituting $x = 0$, $y =$ ……………………, so the answer is ………*

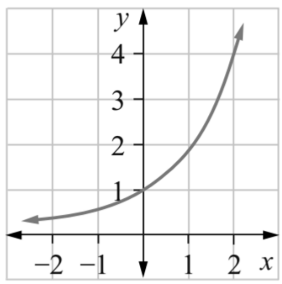
Match the equations with the graphs below:

**a** $y = 2 - x$ | **b** $y = 2x$ | **c** $y = x^2$ | **d** $y = 2^x$



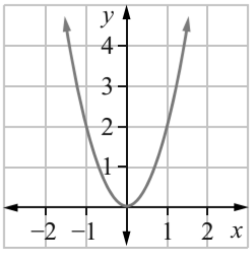



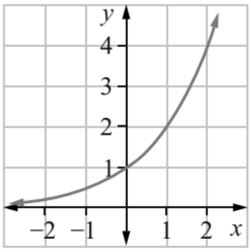



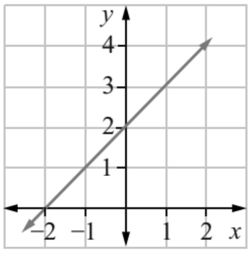



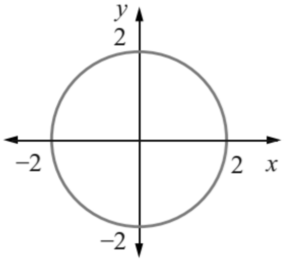
**1** Circle which of the following could be the graph of:

**a** $y = 2^x$

A


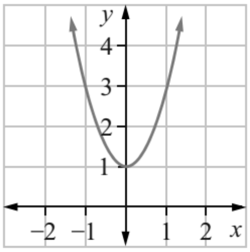

B



C



D



B

**b** $y = 2x^2 + 1$

A
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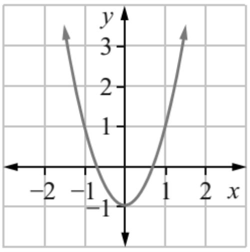


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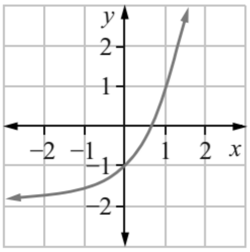


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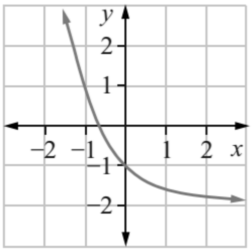
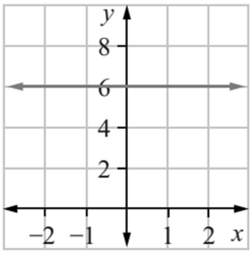



A

**2** Match the equations with the graphs.

| | | |
|---|---|---|
| **a** $y = 2x^2$ .............. | **b** $y = 6$ .............. | **c** $y = x + 1$ .............. |
| **d** $y = x$ .............. | **e** $y = 2^{-x}$ .............. | |

A
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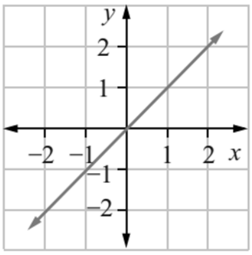


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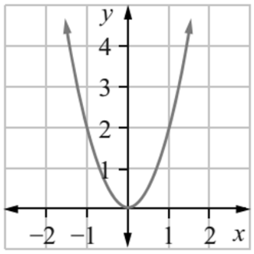


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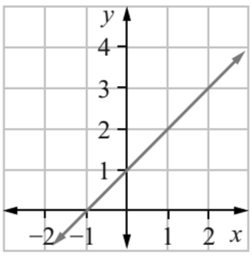
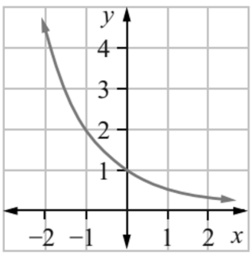
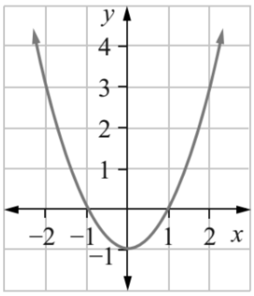



E



a – C, b – A, c – D, d – B, e - E

**3** Match the equations with the graphs.

| | | |
|---|---|---|
| **a** $y = 3^x$ …………… | **b** $y = x^2 + 1$…………… | **c** $y = x + 2$ …………… |
| **d** $y = x^2 - 1$…………… | **e** $y = 1 - x^2$…………… | |

A


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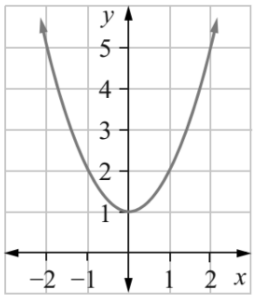

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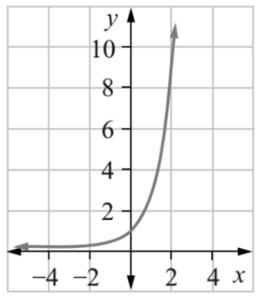

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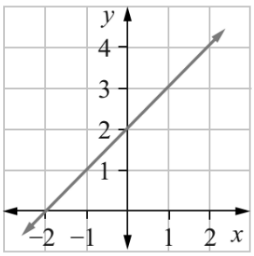

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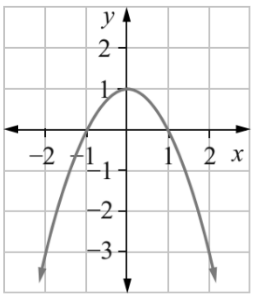
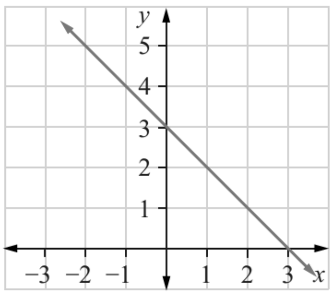


a – C, b – B, c – D, d – A, e - E

**4** Match the equations with the graphs.

| | | |
|---|---|---|
| **a** $y = 3 - x$ …………… | **b** $y = -2x^2$ …………… | **c** $y = 4x^2 - 1$ …………… |
| **d** $y = 5^{-x}$ …………… | **e** $y = 3^x + 1$ …………… | |

A


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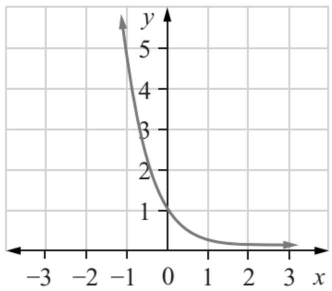

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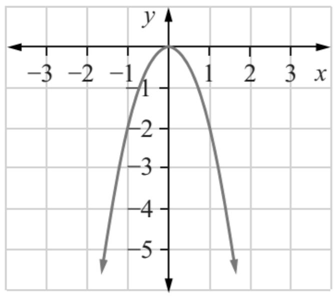

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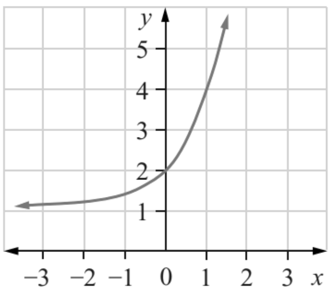
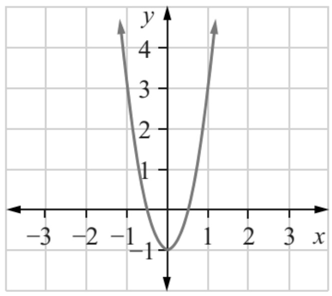
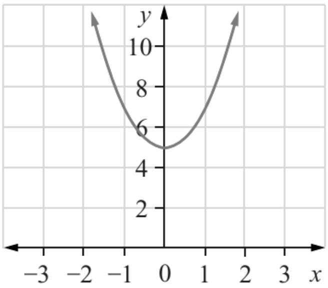


E


a – A, b – C, c – E, d – B, e – D

**5** Match the equations with the graphs.

| | | |
|---|---|---|
| **a** $y = 2x - 4$ .............. | **b** $y = -3x^2$ .............. | **c** $y = 2x^2 + 5$ .............. |
| **d** $y = -5^{-x}$ .............. | **e** $y = 5^x - 1$ .............. | |

A
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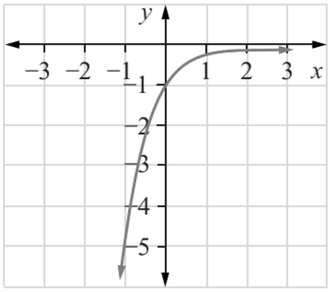


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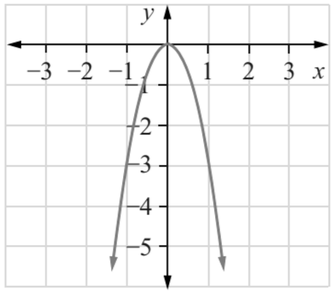
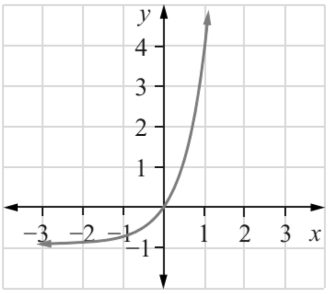
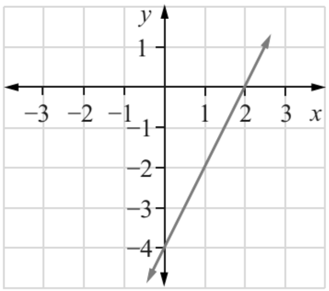


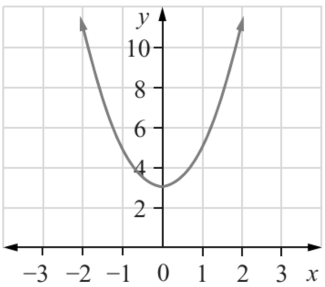

D



E



a – E, b – c, c – A, d – B, e – D

**6** Match the equations with the graphs.

| | | |
|---|---|---|
| **a** $x + 2y - 4 = 0$ ........... | **b** $y = 2x^2 + 3$ .............. | **c** $y = -2x^2 - 3$.............. |
| **d** $y = -2^{-x} + 1$.............. | **e** $y = -2^x - 1$ .............. | |

A
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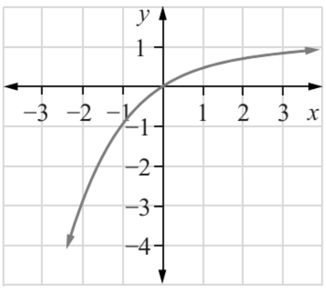


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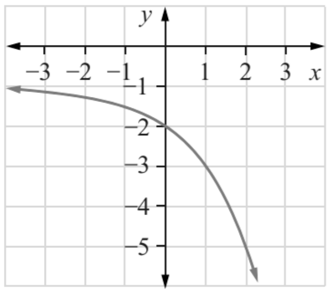


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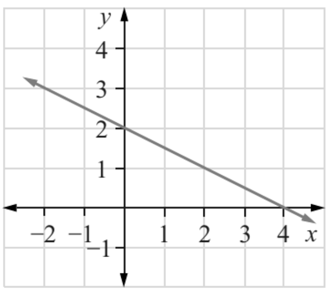


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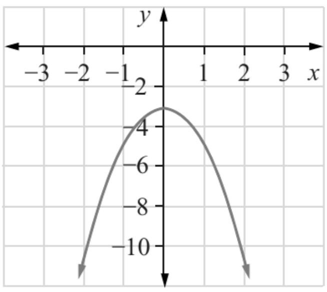
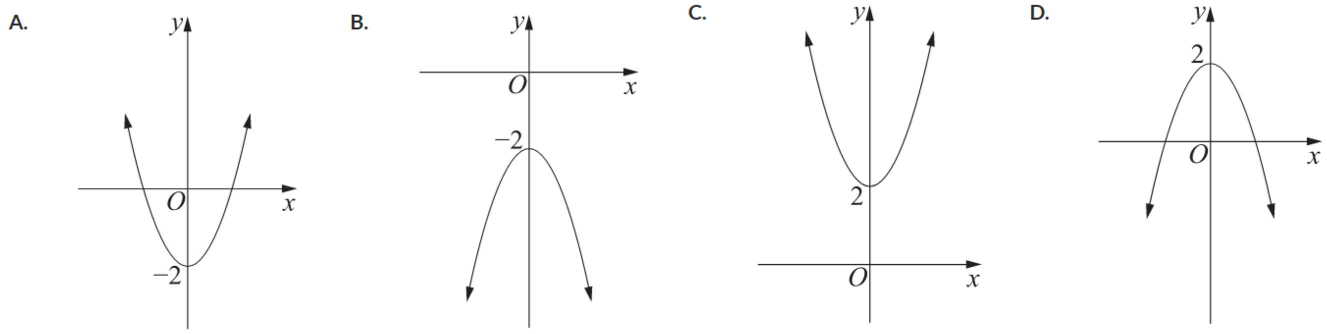



a – D, b – A, c – E, d –B, e – C

**7 2016 HSC Standard 2 Band 3**

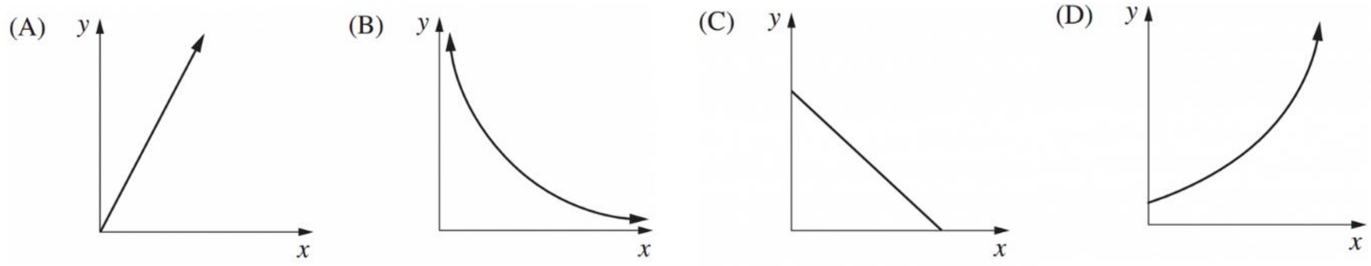
Which graph best represents the equation $y = x^2 - 2$?




B

**8 2008 HSC Standard 2 Band 4**

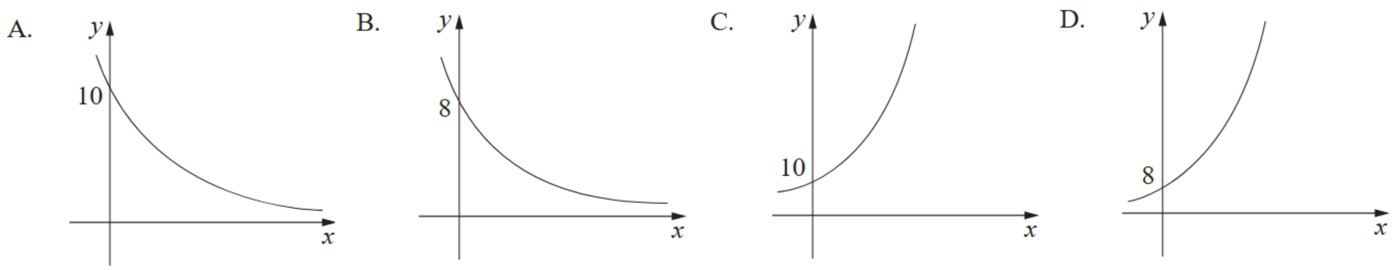
Which graph best represents $y = 3^x$?




D

**9 2021 HSC Standard 2 Band 5**

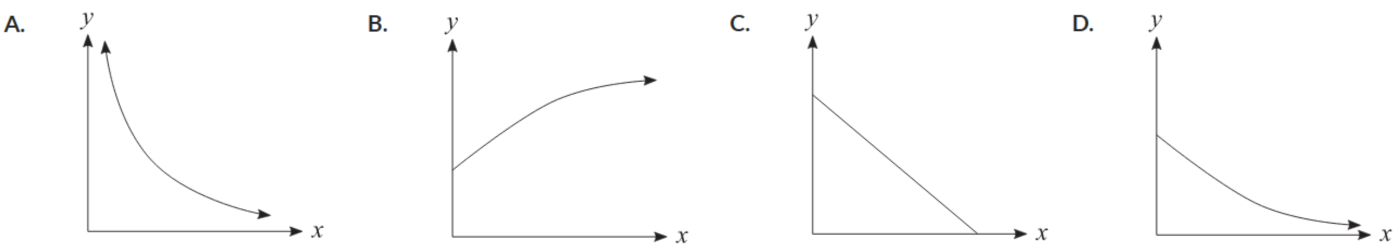
Which of the following best represents the graph of $y = 10(0.8)^x$?




A

**10 HSC Standard 2 Sample Question Band 5**

Which graph best represents $y = 5^{-x}$?

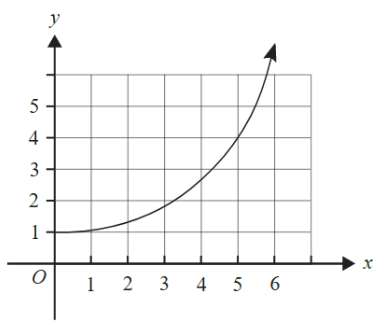



D

**11 HSC Standard 2 Sample Question Band 5**

The graph shows $y = a^x$.

What is the approximate value of $2(a^5)$?




$a^5 = 4$

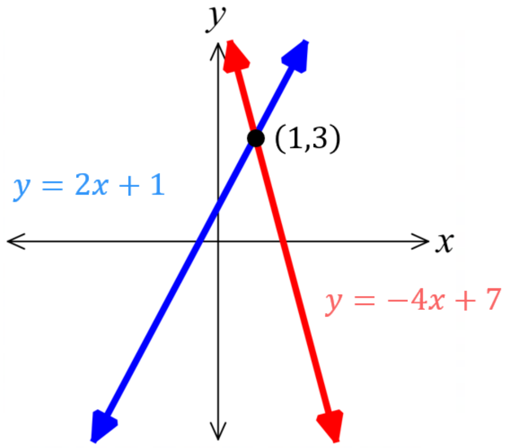
$2a^5 = 8$

# POINTS OF INTERSECTION OF A PARABOLA AND A LINE

- When finding where a parabola meets a line:
  - You have two equations like $y = ax^2 + bx + c$ and $y = mx + c$
  - The intersection point is where both lines have the same $x$ and $y$-values.
  - Find the point of intersection by making the equations equal to each other and solving.

**Example** Find points of intersection.

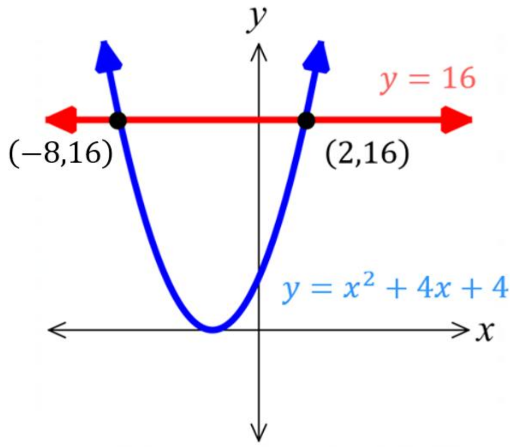
The point of intersection of
$y = 2x + 1$ and $y = -4x + 7$
is $(1,3)$.




Show that this is true by solving:

$$2x + 1 = -4x + 7$$

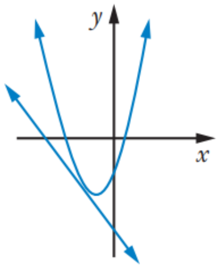
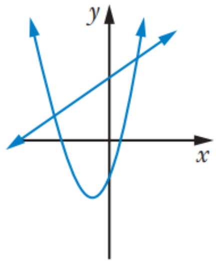
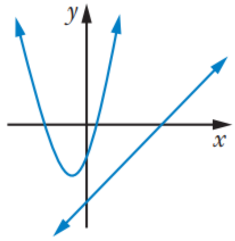
The points of intersection of
$y = x^2 + 4x + 4$ and $y = 16$
are $(2,16)$ and $(-2,16)$




Show that this is true.

## INTERSECTION OF A PARABOLA AND A LINE

- A parabola intersects a straight line at **zero**, **one**, or **two** points.
- The discriminant of the resulting equation after setting the two equations equal reveals the number of points of intersection.

| Discriminant and Points of Intersection | | |
|---|---|---|
|   $\Delta = 0$ |   $\Delta > 0$ |   $\Delta < 0$ |

- To find the points of intersection of a parabola and line:

1. Equate the two equations $y = y$.
2. Rearrange to make a quadratic equation $ax^2 + bx + c = 0$
3. Solve for $x$-coordinates of point of intersection, if they exist.
4. Find $y$-coordinates by substituting $x$-coordinates into one of the original equations.

**Example** Find points of intersection of a line and a parabola.

Find the points of intersection of the line $y = x - 1$ and the parabola $y = x^2 + 4x + 1$.

$y = x - 1$ (1)

$y = x^2 + 4x + 1$ (2)

1. Equate:

Let $(1) = (2)$

$x - 1 = x^2 + 4x + 1$

2. Rearrange:

$0 = x^2 + 3x + 2$

3. Solve for $x$:

$0 = (x + 2)(x + 1)$

$x = -2, -1$

4. Find $y$:

Substitute $x$ values into $(1)$ to find $y$ values:

When $x = -2$: $y = (-2) - 1 = -3$

When $x = -1$: $y = (-1) - 1 = -2$

∴ the two points of intersection are:

$(-2, -3)$ and $(-1, -2)$

Why do we equate the two equations to find the point of intersection?

*At the point of intersection, the …… values will be the same, so we make $y = y$.*

Check the discriminant of $x^2 + 3x + 2 = 0$.

What does this tell you?

$\Delta =$ ………………………

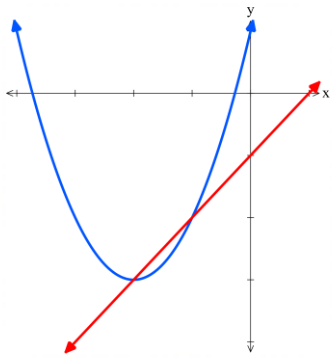
*There are ……… points of intersection.*

Why didn't we check the discriminant of the original parabola $x^2 + 4x + 1 = 0$?

*This will tell you how many points of intersection of the original parabola with the …… axis, i.e. how many $x$-………………*

A sketch is provided here.

Does the solution match the sketch?



Find the points of intersection for the given parabola and line. A sketch has been provided.

**a** $y = x^2 + 2x$ and $y = 5$

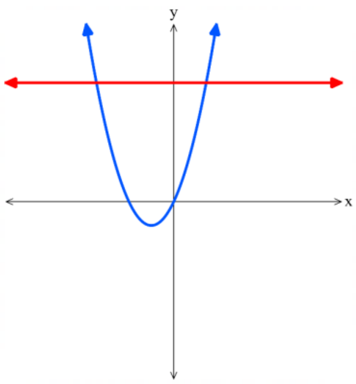
Equate:

$x^2 + 2x = 5$

Rearrange:

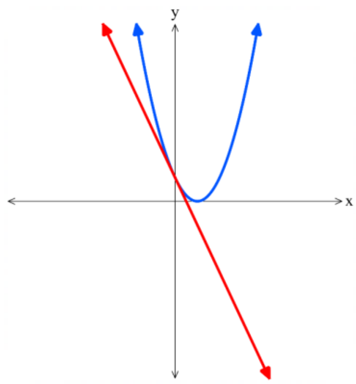
Solve for $x$:

Find $y$:



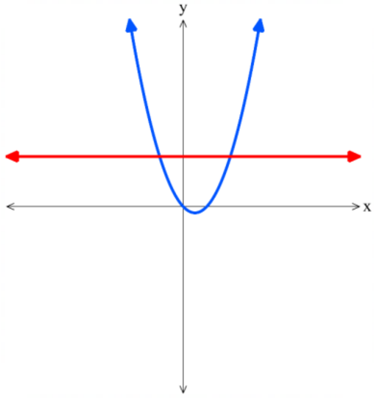

$\left(-1-\sqrt{6}, 5\right)$ and $\left(-1+\sqrt{6}, 5\right)$

**b** $y = x^2 - 2x + 1$ and $y = -2x + 1$




$(0,1)$

**1** The graphs of $y = x^2 - x$ and $y = 2$ are shown. Solve the two equations simultaneously for $x$.




Equate:

$x^2 - x = 2$

Rearrange:

Solve for $x$:

$(-1,2)$ and $(2,2)$

Do your solutions roughly match what is shown on the graph? ..............

yes

**2** The graph of a parabola $y = x^2 - x - 1$ is shown.

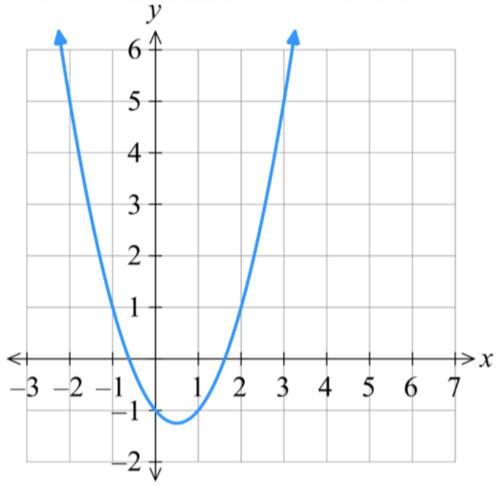
**a** Sketch the line $y = x + 2$ on this graph.

**b** Use the grid to find the points of intersection.

.....................................................

**c** Find the points of intersection algebraically:

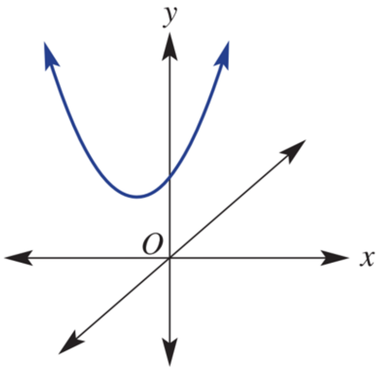
$x^2 - x - 1 = x + 2$




$(-1,1)$ and $(3,5)$

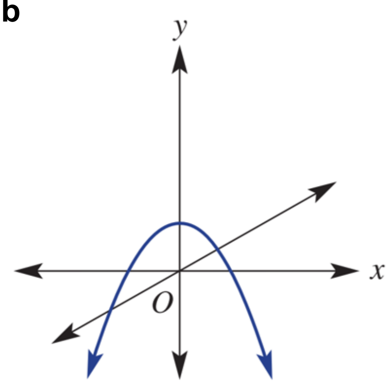
**3** What would be the value of the discriminant $\Delta = b^2 - 4ac$ from solving the following equations simultaneously? Answer with $\Delta < 0$, $\Delta = 0$, or $\Delta > 0$.

**a**



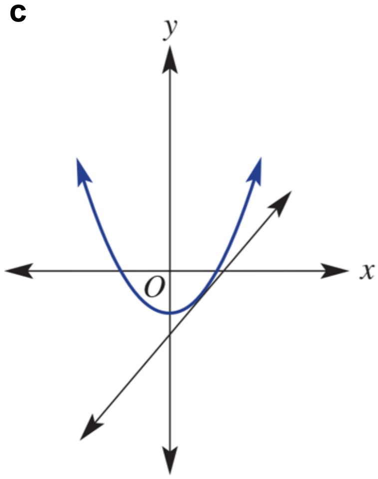

$\Delta < 0$

**b**




$\Delta > 0$

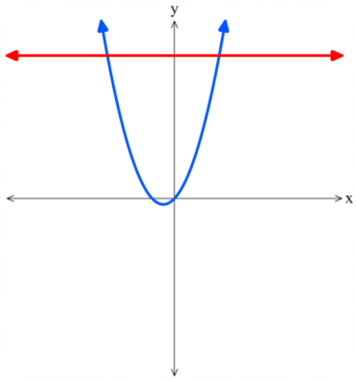
**c**




$\Delta = 0$

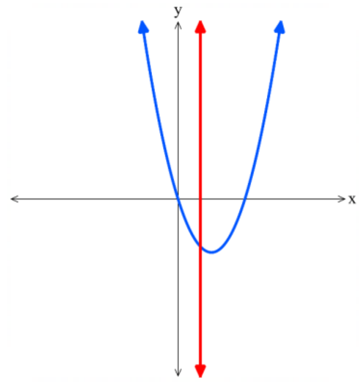
**4** Find the points of intersection for these lines and parabolas.

**a** $y = x^2 + x$ and $y = 6$



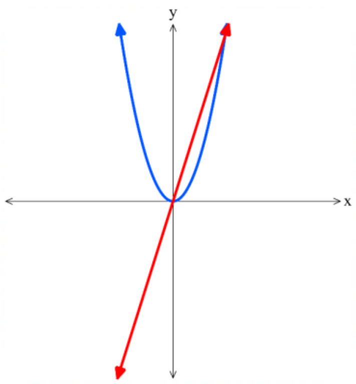

$(-3,6)$ and $(2,6)$

**b** $y = x^2 - 3x$ and $x = 1$



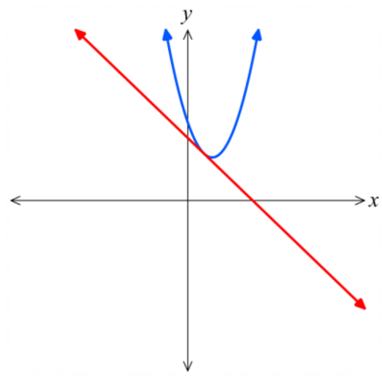

$(1,-2)$

**c** $y = x^2$ and $y = 3x$



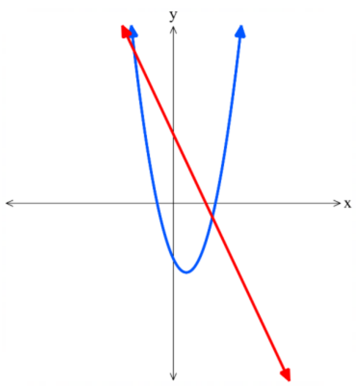

$(0,0)$ and $(3,9)$

**d** $y = x^2 - 3x + 5$ and $y = -x + 4$



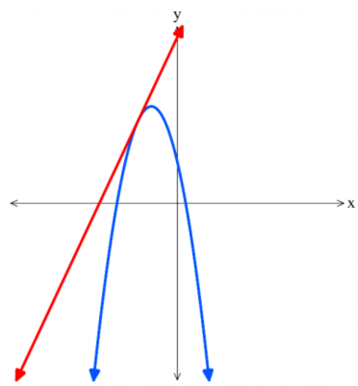

$(1,3)$

**e** $y = x^2 - 2x - 4$ and $y = -2x + 5$




$(-3,11)$ and $(3,-1)$

**f** $y = -x^2 - 4x + 3$ and $y = 2x + 12$




$(-3,6)$

**5** Solve the following simultaneous equations, making use of the quadratic formula. Leave answers in exact surd form. Check your answer using DESMOS.

**a** Find the point of intersection of $y = 2x^2 + 3x + 6$ and $y = x + 4$

No point of intersection.

**b** Find the point of intersection of $y = x^2 + 2x - 5$ and $y = x$

$\left(\frac{-1-\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2}\right)$ and $\left(\frac{-1+\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right)$

**6** A member of an indoor cricket team, playing a match in a gymnasium, hits a ball that follows a path given by $y = -0.1x^2 + 2x + 1$ , where $y$ is the height above ground, in metres, and $x$ is the horizontal distance travelled by the ball. The ceiling of the gymnasium is 10.6 metres high. Determine whether the ball will intersect with the roof.

Set up equation $10.6 = -0.1x^2 + 2x + 1$

Yes, as $\Delta > 0$.

**7**

**a** Use the discriminant to show that the line $y = 2x + 1$ does not intersect the parabola $y = x^2 + 3$.

$\Delta = -4$, so no solutions.

**b** Determine for which values of $k$ the line $y = 2x + k$ intersects the parabola $y = x^2 + 3$.

$$2x + k = x^2 + 3$$
$$x^2 - 2x + 3 - k = 0$$

$$\begin{aligned}\Delta &= 4 - 4(1)(3 - k)\\ &= 4 - 12 + 4k\\ &= 4k - 8\end{aligned}$$

$$\begin{aligned}4k - 8 &\geq 0\\ k &\geq 2\end{aligned}$$

**8** The line $y = mx$ is a **tangent** to the parabola $y = x^2 - 2x + 4$
(i.e the line touches the parabola at just one point).

**a** Find the possible values of $m$.

Set up equation $mx = x^2 - 2x + 4$

$x^2 + (-2 - m)x + 4 = 0$

Tangent means 1 intersection point. $b^2 - 4ac = 0$

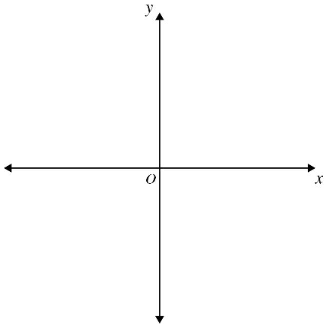
$(-2 - m)^2 - 4(1)(4) = 0$

$m^2 + 4m - 12 = 0$

$(m + 6)(m - 2) = 0$

$m = 2 \text{ or } -6$

**b** Explain why there are two possible values of $m$.
A diagram may help (use DESMOS to sketch $y = x^2 - 2x + 4$ and $y = mx$).




There are two ways to draw a line that is tangent to $y = x^2 - 2x + 4$,
one with gradient $-6$ and the other with gradient $2$

**c** If the value of m is changed so that the line now intersects the parabola in two places, what is the set of possible values of m?

$\Delta > 0$

$(m + 6)(m - 2) > 0$

By sketching the parabola $y = (x + 6)(x - 2)$ and solving the inequality, $m < -6, m > 2$

**9** The line $y = mx - 4$ is tangent to the parabola $y = x^2 - 2x + 5$. Find the values of $m$.

4 or $-8$

**10** Find the values of $p$ for which $y = px - 3p - 5$ is tangent to $x^2 = 8y$.

$p = \frac{5}{2}$ or $p = -1$

# POINTS OF INTERSECTION OF TWO GRAPHS

- For any two functions $y = f(x)$ and $y = g(x)$, set $f(x) = g(x)$ and solve to find the $x$-coordinate(s) of the point of intersection.

**Example** Find points of intersection of a line and a hyperbola.

Find the points of intersection of the line $y = 4x$ and the hyperbola $y = \frac{1}{x}$

1. Solve simultaneous equation for $x$ values:

$y = 4x$ (1)

$y = \frac{1}{x}$ (2)

Let $(1) = (2)$

$4x = \frac{1}{x}$

$4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

2. Find $y$ values by substitution:

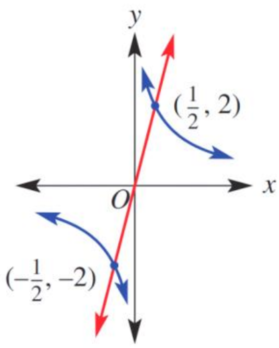
Substitute $x$ into $(1)$ to find $y$ values:

When $x = \frac{1}{2}$ $\quad y = 4\left(\frac{1}{2}\right) = 2$

When $x = \frac{-1}{2}$ $\quad y = 4\left(\frac{-1}{2}\right) = -2$

$\therefore$ the two points of intersection are:

$\left(\frac{1}{2}, 2\right)$ and $\left(-\frac{1}{2}, -2\right)$




Find the point of intersection of these graphs. Check your answer using DESMOS.

**a** $y = 3$ and $y = \frac{2}{x}$

$\left(\frac{2}{3}, 3\right)$

**b** $y = \frac{1}{x}$ and $y = 2x$

$\left(-\frac{1}{\sqrt{2}}, -\sqrt{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$

**c** $y = -\frac{2}{x}$ and $y = -2x$

$(-1, 2)$ and $(1, -2)$

**Example** Find points of intersection of a line and a circle.

Find the points of intersection of the line $y = 2x$ and the circle $x^2 + y^2 = 4$

1. Solve simultaneous equation for $x$ values:

$y = 2x$ (1)

$y^2 + x^2 = 4$ (2)

Sub (1) into (2)

$$(2x)^2 + x^2 = 4$$
$$5x^2 = 4$$
$$x^2 = \frac{4}{5}$$
$$x = \pm\frac{2}{\sqrt{5}}$$

2. Find $y$ values by substitution:

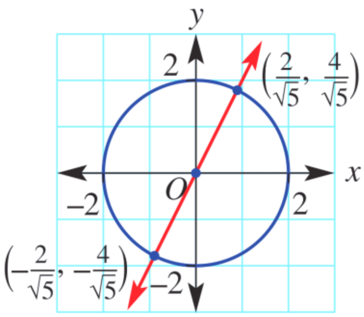
Substitute into (1) to find $y$ values:

When $x = \frac{2}{\sqrt{5}}$ $\quad y = 2\left(\frac{2}{\sqrt{5}}\right) = \frac{4}{\sqrt{5}}$

When $x = \frac{-2}{\sqrt{5}}$ $\quad y = 2\left(\frac{-2}{\sqrt{5}}\right) = \frac{-4}{\sqrt{5}}$

∴ the two points of intersection are:

$\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$ and $\left(\frac{-2}{\sqrt{5}}, \frac{-4}{\sqrt{5}}\right)$




Find the point of intersection of these graphs. Check your answer using DESMOS.

**a** $x^2 + y^2 = 9$ and $y = x$

**b** $x^2 + y^2 = 16$ and $y = 4x$

**c** $y = -x + 1$ and $x^2 + y^2 = 5$

$\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)$ and $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

$\left(-\frac{4}{\sqrt{17}}, -\frac{16}{\sqrt{17}}\right)$ and $\left(\frac{4}{\sqrt{17}}, \frac{16}{\sqrt{17}}\right)$

$(-1,2)$ and $(2,-1)$

**1** Find the point of intersection of these graphs. You will need to rearrange before equating. Check your answer using DESMOS.

**a** $y = x^2$ and $x = \frac{y}{2}$

$(0,0)$ and $(2,4)$

**b** $y + 4x^2 + x - 6 = 0$ and $y - 3x = 7$

$(-0.5, 5.5)$

**c** $y = x^2 + 3x - 1$ and $3y = x - 3$

$\left(-\frac{8}{3}, -\frac{17}{9}\right)$ and $(0, -1)$

**d** $y - 7 = (x - 2)^2$ and $y + x = 9$

$(1,8)$ and $(2,7)$

**e** $xy = 1$ and $y = x - 1$

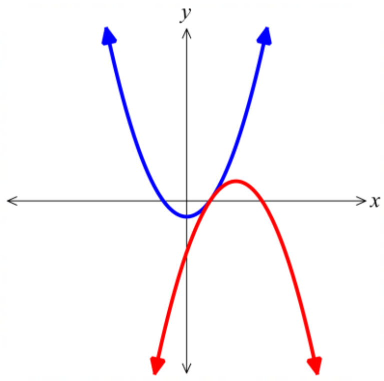
$\left(\frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right)$ and $\left(\frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}\right)$

**f** $x^2 + y^2 = 4$ and $2y + x = 0$

$\left(\frac{-4}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{4}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$

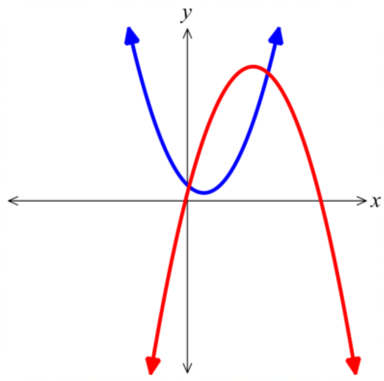
**2** Find the point of intersection for these functions.
A sketch has been provided to help you interpret your answer.

**a** $y = x^2 - 2$
$y = -x^2 + 6x - 6.5$



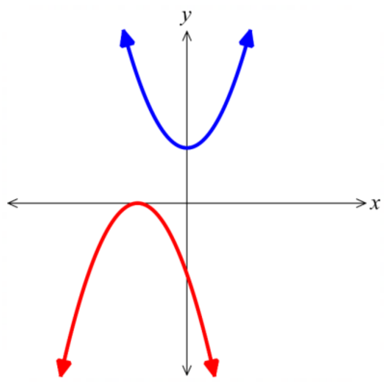

(1.5, 0.25)

**b** $y = x^2 - 2x + 2$
$y = -x^2 + 8x + 1$



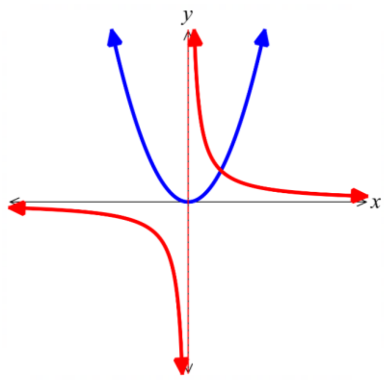

$\left(\frac{5-\sqrt{23}}{2}, 9 - \frac{3\sqrt{23}}{2}\right)$ and $\left(\frac{5+\sqrt{23}}{2}, 9 + \frac{3\sqrt{23}}{2}\right)$

**c** $y = x^2 + 7$
$y = -x^2 + 6x - 9$




You will end up with a quadratic where $\Delta < 0$,
no points of intersection

**d** $y = x^2$
$y = \dfrac{8}{x}$




(2,4)

**3** Show that $y = -2x + 5$ is tangent to the circle $x^2 + y^2 = 5$.

Equate to find points of intersection:

$x^2 + (-2x + 5)^2 = 5$

$x^2 + 4x^2 - 20x + 25 = 5$

$5x^2 - 20x + 20 = 0$

$\Delta = b^2 - 4ac = (-20)^2 - 4(5)(20) = 0$

$\therefore$ there is 1 point of intersection, hence the line is tangent to the circle.

**4** Show that $3x + y = 6$ is tangent to the circle $x^2 + y^2 + 4x + 16y + 28 = 0$.

Same method as above.

$x^2 + (-3x + 6)^2 + 4x + 16(-3x + 6) + 28 = 0$

This simplifies to: $x^2 - 8x + 16 = 0$

$\Delta = b^2 - 4ac = (-8)^2 - 4(1)(16) = 0$

**5** The line $2x - y + 3 = 0$ intersects the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ at two points, $D$ and $E$.

**a** Find the coordinates of $D$ and $E$.

Rearrange equation of line: $y = 2x + 3$

Sub into equation of circle

$x^2 + (2x + 3)^2 - 4x + 6(2x + 3) - 12 = 0$

This simplifies to $5x^2 + 20x + 15 = 0$

$\therefore (x + 1)(x + 3) = 0$

The points of intersection are $(-1, 1), (-3, 3)$

**b** Hence find the length of the line segment joining $DE$.

$DE = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{2^2 + 4^2} = 2\sqrt{5}$ units

**6** For the circle $x^2 + y^2 = 4$ and line $y = mx + 4$ , determine the exact values of the gradient, $m$, so that the line:

**a** is a tangent to the circle (intersects the circle once)

set up equation $x^2 + (mx + 4)^2 = 4$

$\Delta = 16m^2 - 48$

Let $\Delta = 0$

$m = \pm\sqrt{3}$

**b** intersects the circle at two points

$\Delta > 0$

$16m^2 - 48 > 0$

$m < \sqrt{3}$ or $m > \sqrt{3}$

**c** does not intersect the circle

$-\sqrt{3} < m < \sqrt{3}$

**7**

**a** Try to find the coordinates of the points of intersection of $y = \frac{1}{x}$ and $y = -x + 1$.

What do you notice?

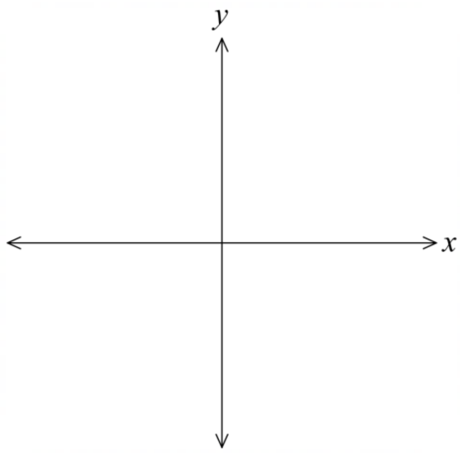
$-x + 1 = \frac{1}{x}$

$-x^2 + x = 1$

$x^2 - x + 1 = 0$

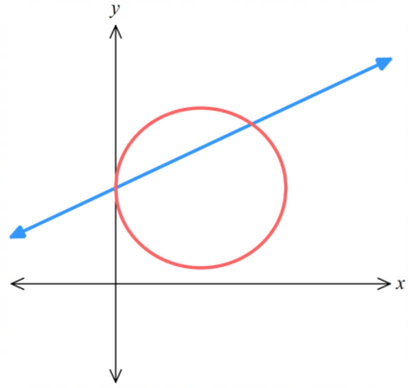
There are no points of intersection; $\Delta = (-1)^2 - 4(1)(1) = -3 < 0$.

**b** Sketch the two graphs on the same axis to visualise the scenario.



**8** The line $2y - x = 12$ intersects the circle $x^2 + y^2 - 10x - 12y + 36 = 0$ at the points $A$ and $B$.
A sketch is given, it would be useful to add to it as you go through parts a to d.

**a** Find the coordinates of $A$ and $B$.




Using same method as previous question,

$(0,6)$ and $(8,10)$

**b** Find the equation of the perpendicular bisector of $AB$.

The perpendicular bisector of AB is the perpendicular line passing through the midpoint of AB.

Midpoint of AB: $\left(\frac{0+8}{2}, \frac{6+10}{2}\right) = (4,8)$

Gradient of AB: $m = \frac{10-6}{8-0} = \frac{1}{2}$

$\therefore$ gradient of perpendicular bisector is $-2$

Using $m = -2$ and $(4,8)$: $y - y_1 = m(x - x_1)$

$y - 8 = -2(x - 4)$

$y = -2x + 16$

**c** The perpendicular bisector of $AB$ intersects the circle at the points $P$ and $Q$.
Find the exact coordinates of $PQ$.

Using same method as part **a**, $(5 + \sqrt{5}, 6 - 2\sqrt{5})$ and $(5 - \sqrt{5}, 6 + 2\sqrt{5})$

**d** Explain why APBQ is a kite and hence find its area.

The diagonals meet at right angles, so it satisfies the definition of a kite.

Using distance formula:

Length of AB is: $4\sqrt{5}$

Length of PQ is: $10$

$A = \frac{xy}{2} = \frac{4\sqrt{5}(10)}{2} = 20\sqrt{5}$

**9** Mr Ding used ChatGPT to generate an exam question:
'Find the points of intersection between $y = 2^x$ and $y = x^2 + 1$'
He checked on DESMOS and saw two 'nice' points of intersection at $(0,1)$ and $(1,2)$.
However, his students would not be able to find these points algebraically. Explain why.

It is not possible to solve $2^x = x^2 + 1$ using any algebraic techniques (and none exist!)

**10** [Auxiliary circle of a hyperbola]
The equation $(x + y)^2 + (x - y)^2 = a^2$ is an equation representing a circle.

**a** By expanding then completing the square, find the radius of a circle in this form, in terms of $a$.

$r = \frac{a}{\sqrt{2}}$

**b** What kind of curve is $(x + y)^2 - (x - y)^2 = a^2$? (hint: expand and simplify)

Hyperbola with equation $y = \frac{a^2}{4x}$

**c** Use DESMOS to sketch $(x + y)^2 + (x - y)^2 = a^2$ and $(x + y)^2 - (x - y)^2 = a^2$.
Create a slider for $a$ and experiment with values of $a$.
https://www.desmos.com/calculator/luwgidnzzy
What do you think the coordinates of the point of intersection are?

The points of intersection are $\left(\pm\frac{a}{2}, \pm\frac{a}{2}\right)$

**d** Verify this algebraically.

Start by expanding out each equation, after fully simplifying you should have $2x^2 + 2y^2 = a^2$ and $4xy = a^2$

Substitute $y = \frac{a^2}{4x}$ into the first equation to get $32x^4 - 16a^2x^2 + 2a^4 = 0$

Divide by the common factor and recognise as a perfect square: $(4x^2 - a^2)^2 = 0$

Solve to get $x = \pm\frac{a}{2}$

Substitute into a previous equation and solve for $y$: $y = \pm\frac{a}{2}$