Student Name

Mathematics Stage 5 Paths

POLYNOMIALS

Book 1 Define and operate with polynomials

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OVERVIEW

SYLLABUS CONTENT

MA5-POL-P-01 defines, operates with and graphs polynomials and applies the factor and remainder theorems to solve problems

Define and operate with polynomials

- Recognise a polynomial expression $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ where $n = 0,\ 1,\ 2$ ... and $a_0,\ a_1,\ a_2, \ldots, a_n$ are real numbers
- Describe polynomials using terms such as degree, leading term, coefficient and leading coefficient, constant term, monic and non-monic
- Define a monic polynomial as having a leading coefficient of one
- Apply the notation $P(x)$ for polynomials and $P(c)$ to indicate the value of $P(x)$ for $x = c$
- Add, subtract and multiply polynomials

INTRODUCTION TO POLYNOMIALS

POLYNOMIALS

- **Polynomials** are algebraic expressions with many terms and positive powers.
  - **Binomial**: $2x + 3$
  - **Trinomial**: $3x^2 + 5x + 12$
  - **Polynomial**: $7x^{12} - 5x^9 + 3x^3 + 5x - 1$

- Polynomials are named by the highest power of x. This is called the **degree** of the polynomial.
  - **Constant**: 2
  - **Linear**: $3x - 7$
  - **Quadratic**: $2x^2 - 4x + 11$
  - **Cubic**: $-4x^3 + 6x^2 - x + 3$
  - **Quartic**: $x^4 - x^3 - x^2 - 2$
  - **Degree 8**: $3x^8 - 4x^5 - 3$

Example Identify polynomials.

Determine whether the following are polynomials. If so, state the name of the polynomial.

a $-x^2 + 5x - 4$ Quadratic polynomial.	**b** $x^3 + x^2 + \sqrt{x} + 3$ Not a polynomial.	**c** $7x^5 + 2x^4 - 2$
d $2x^2 - \sqrt{x} + 13$	**e** $x^3 + \frac{2}{x} + 5$	**f** $\frac{1}{5}x^3 + 3x + 13$
g $x^{\frac{1}{3}} - 5x^2 - 51x - 2$	**h** 3	**i** $2 - x^{255}$

FORMAL DEFINITION OF A POLYNOMIAL

- A polynomial is a function of the form:

Degree (highest power)

$$P(x) = \overbrace{a_n}^{\text{Leading coefficient}}x^{n} + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

$$\underbrace{a_nx^n}_{\text{Leading term (term with the highest power)}}$$

Constant term

The powers are positive whole numbers, including 0.

The coefficients a_0, a_1, a_2 ... are real numbers.

- A **monic** polynomial has leading coefficient of 1.

Example Describe polynomials.

$P(x) = x^3 - 3x^2 - x + 2$		$P(x) = 5x^2 + 2x^6 - 1$	
Degree:	3	Degree:	6
Leading term:	x^3	Leading term:	$2x^6$
Leading coefficient:	1	Leading coefficient:	2
Monic/Non-Monic:	Monic	Monic/Non-Monic:	Non-monic
Constant term:	2	Constant term:	-1

Identify the features of these polynomial functions.

a $P(x) = 3x^4 - x^2 + 5x - 1$

Degree:

4

Leading term:

$3x^4$

Leading coefficient:

3

Monic/Non-Monic:

non-monic

Constant term:

-1

b $P(x) = 4 - x + \dfrac{3x^2}{2}$

Degree:

2

Leading term:

$\dfrac{3x^2}{2}$

Leading coefficient:

$\dfrac{3}{2}$

Monic/Non-Monic:

non-monic

Constant term:

4

1 A polynomial expression is given by $3x^4 - 2x^3 + x^2 - x + 2$.

a How many terms does the polynomial have?

5

b State the coefficient of:

x^4	x^3	x^2	x
3	-2	1	-1

c What is the constant term?

2

2 Determine if these polynomials are constant, linear, quadratic, cubic or quartic.

a $2x - 5$	**b** $x^2 - 3$	**c** $x^4 + 2x^3 + 1$	**d** $1 + x + 3x^2$
linear	quadratic	quartic	quadratic
e 6	**f** $3 - \frac{2}{5}x$	**g** -2	**h** $1 - x^4$
constant	linear	constant	quartic

3 State the leading coefficient in each of the following:

a $2x^3 + 4x^2 - 2x$	**b** $5x^4 + 3x^2 - 1$	**c** $2x^2 - x^4 - 2$	**d** $-7x^5 + \frac{x^2}{2} + 8$
2	5	-1	-7

4 Identify the degree of each polynomial.

a $5x^7 - 3x^5 + 2x^3$	**b** $3 + x + x^2 - x^3$	**c** $3x + 5$
7	3	1
d $x^{11} - 5x^8 + 4$	**e** $2 - x - 5x^2 + 3x^3$	**f** 3
11	3	0

5 In each case, state if the function is a polynomial, If so, write down its leading coefficient.

a $f(x) = 2x + 3$	**b** $f(x) = -x^3 - 4x^2 + 5$	**c** $f(x) = 3x^2 - x^{-1}$
Yes, 2	Yes, -1	No
d $f(x) = \sqrt{x} + x - 1$	**e** $f(x) = -\frac{x^3}{6} - x + 1$	**f** $f(x) = \sqrt{x^2 - 9} + 1$
No	Yes, $-\frac{1}{6}$	No

6 Identify the features of these polynomial functions.

a $P(x) = 4x^3 - 2x^2 - 1$

Degree: 3

Leading term: $4x^3$

Leading coefficient: 4

Monic/Non-Monic: Non-monic

Constant term: -1

b $P(x) = 4 - \frac{5x^2}{2}$

Degree: 2

Leading term: $-\frac{5x^2}{2}$

Leading coefficient: $-\frac{5}{2}$

Monic/Non-Monic: Non-monic

Constant term: 4

c $P(x) = 10 - 4x - 6x^3$

Degree: 3

Leading term: $-6x^3$

Leading coefficient: -6

Monic/Non-Monic: Non-monic

Constant term: 10

d $P(x) = x^2(x - 2)$

Degree: 3

Leading term: x^3

Leading coefficient: 1

Monic/Non-Monic: Monic

Constant term: 0

7 A polynomial is a function $P(x) = x^3 - 7x^2 + x - 1$.
Therefore, we can evaluate polynomials using function notation. Evaluate the following:

a $P(2)$ -19

b $P(-1)$ -10

c $P(0)$ -1

8 $P(x) = 2x^4 - 3x^3 + 5x - 4$. Evaluate:

a $P(1)$ 0

b $P(3)$ 92

c $P(-1)$ -4

9 Evaluate $P(-2)$ for

a $P(x) = (x + 2)^2$ 0

b $P(x) = (x - 2)(x + 3)(x + 1)$ 4

c $P(x) = x^2(x + 5)(x - 7)$ -108

10 $P(x) = x^3 - x^2, Q(x) = 4 - 3x$. Evaluate:

a $P(1) + Q(2)$ -2	**b** $P(3) + Q(-1)$ 25	**c** $P(-2) - Q(-2)$ -22
d $Q(1) - P(3)$ -17	**e** $\left(P(2)\right)^2 + \left(Q(1)\right)^2$ 17	**f** $\left(P(-1)\right)^3 - \left(Q(-1)\right)^3$ -351

11 Identify the leading term of these polynomials.

a $P(x) = \frac{4-2x^2}{4}$ $-\frac{1}{2}x^2$	**b** $P(x) = \frac{x^3+7x^2+x-3}{-7}$ $-\frac{x^3}{7}$	$P(x) = \frac{x^3+4x^2}{-8}$ $-\frac{x^3}{8}$
c $P(x) = (x+2)^2$ x^2	**d** $P(x) = -(x+2)(x+3)^2$ $-x^3$	$P(x) = 2x(x+5)^2(7-x)$ $-2x^4$
e $P(x) = (x^2 - 3x)(1 - x^3)$ $-x^5$	**f** $P(x) = (2x^3 - 3)^3$ $8x^9$	

12 For the polynomial $P(x) = (a+1)x^3 + (b-7)x^2 + c + 5$, find a, b, c given that:

The degree is 2, $P(x)$ is monic, and the constant term is -1

$a + 1 = 0, b - 7 = 1, c + 5 = -1$

$a = -1, b = 8, c = -6$

13 These expressions are equivalent. By expanding and comparing coefficients, find the values of the pronumerals.

a $(a-b)x^2 + (2a+b)x = 7x - x^2$

$a = 2, b = 3$

b $a(x-1)^2 + b(x-1) + c = x^2$

$a = 1, b = 2, c = 1$

OPERATIONS WITH POLYNOMIALS

ADDITION AND SUBTRACTION

- Add and subtract two polynomials by combining like terms.
- Ensure you write brackets, especially when subtracting.

Example Add and subtract polynomials.

For $P(x) = x^2 - x + 3$ and $Q(x) = 2x^2 + x - 1$

Find $P(x) + Q(x)$

$= (x^2 - x + 3) + (2x^2 + x - 1)$

$= 3x^2 + 2$

Find $P(x) - Q(x)$

$= (x^2 - x + 3) - (2x^2 + x - 1)$

$= x^2 - x + 3 - 2x^2 - x + 1$

$= -x^2 - 2x + 4$

Why should we always use brackets when substituting polynomials?

Brackets help us avoid mistakes when ....

For $P(x) = x^3 - x^2$ and $Q(x) = 4 - 3x$:

Find $P(x) + Q(x)$

$x^3 - x^2 - 3x + 4$

Find $P(x) - Q(x)$

$x^3 - x^\# + 3x - 4$

For $P(x) = x^2 - 3$ and $Q(x) = x^3 - x^2 + x$:

Find $3P(x) + Q(x)$

$x^3 + 2x^2 + x - 9$

Find $P(x) - 2Q(x)$

$-2x^3 + 3x^2 + 2x - 3$

POLYNOMIAL MULTIPLICATION – AREA MODEL

Example Multiply polynomials.

$(x^2+1)(x^3-x+1)$

	x^3	$-x$	1
x^2	x^5	$-x^3$	x^2
1	x^3	$-x$	1

$= x^5+x^2-x+1$

$(x^2+x+1)(x^3+2x-3)$

	x^3	$2x$	-3
x^2	x^5	$2x^3$	$-3x^2$
x	x^4	$2x^2$	$-3x$
1	x^3	$2x$	-3

$= x^5+x^4+3x^3-x^2-6$

Expand these expressions using the area model.

a $(x-4)(x^2+4)$

$x^3-4x^2+4x-16$

b $(x+16)(x^2+5x+2)$

$x^3+21x^2+82x+32$

c $(x^5+2x^2+3)(x^2+4x+5)$

$x^7+4x^6+5x^5+2x^4+8x^3+13x^2+12x+15$

d $(x^4+16x^2+2x)(5x^2-2x+5)$

$5x^6-2x^5+85x^4-22x^3+76x^2+10x$

POLYNOMIAL MULTIPLICATION – DISTRIBUTIVE METHOD

- Like expanding binomials, we multiply every term of one polynomial by every term of the other polynomial.

$$(x^2 + 1)(x^3 - x + 1)$$

Example Multiply polynomials.

$(x^2 + 1)(x^3 - x + 1)$	$(x^2 + x + 1)(x^3 + 2x - 3)$
$= x^2(x^3 - x + 1) + 1(x^3 - x + 1)$	$= x^2(x^3 + 2x - 3) + x(x^3 + 2x - 3) + 1(x^3 + 2x - 3)$
$= x^5 - x^3 + x^2 + x^3 - x + 1$	$= x^5 + 2x^3 - 3x^2 + x^4 + 2x^2 - 3 + x^3 + 2x - 3$
$= x^5 + x^2 - x + 1$	$= x^5 + x^4 + 3x^3 - x^2 - 6$

Expand these expressions.

a $(2x^2 + 1)(3x + 4x - 3)$

$14x^3 - 6x^2 + 7x - 3$

b $(2x - 3)(x^2 - 5x + 1)$

$2x^3 - 13x^2 + 17x - 3$

$P(x) = x^2 + x - 1$ and $Q(x) = x^3 + 2x + 3$

a Simplify $P(x) \times Q(x)$

$x^5 + x^4 + x^3 + 5x^2 + x - 3$

b Simplify $P(x)^2$

$x^4 + 2x^3 - x^2 - 2x + 1$

1 Find the sum $P(x) + Q(x)$ and difference $P(x) - Q(x)$ of these polynomials.

a $P(x) = 3x^2 - 5x + 2$ $Q(x) = x^2 + 7x - 3$

$P(x) + Q(x)$

$4x^2 + 2x - 1$

$P(x) - Q(x)$

$2x^2 - 12x + 5$

b $P(x) = x^3 - 2x^2 + 3x + 7$ $Q(x) = 2x^3 + 3x^2 - 5x + 2$

$P(x) + Q(x)$

$3x^3 + x^2 - 2x + 9$

$P(x) - Q(x)$

$-x^3 - 5x^2 + 8x + 5$

c $P(x) = x^4 + 9x^3 + 7x^2 - 4x + 9$ $Q(x) = 4x^3 - 8x^2 - 15$

$P(x) + Q(x)$

$x^4 + 13x^3 - x^2 - 4x - 6$

$P(x) - Q(x)$

$x^4 + 5x^3 + 15x^2 - 4x + 24$

d How is the degree of $P(x) \pm Q(x)$ related to the degree of $P(x)$ and $Q(x)$ for these polynomials?

the degree of $P(x) \pm Q(x)P(x)$ is equal to the maximum of the degrees of P(x) and Q(x).

e Will your answer to part **d** be true for all polynomials $P(x)$ and $Q(x)$?
(consider what happens if the leading terms of $P(x)$ and $Q(x)$ are opposites.)

No.

For $P(x) + Q(x)$, this won't be the case if both polynomials have the same degree and their leading coefficients sum to zero.

For $P(x) - Q(x)$, this won't be the case if both polynomials have the same degree and their leading coefficients are equal.

14 Given $P(x) = x + 5$ and $Q(x) = 2x - 1$, find:

a $P(3)$

8

b $Q(-2)$

-5

c $P(3) + Q(-2)$

3

d The degree of $P(x) + Q(x)$

1

e The degree of $P(x)Q(x)$

2

2 If $P(x) = 5x + 2$ and $Q(x) = x^2 - 3x + 1$, find:

a $P(x) + Q(x)$

$x^2 + 2x + 3$

b $Q(x) + P(x)$

$x^2 + 2x + 3$

c $P(x) - Q(x)$

$-x^2 + 8x + 1$

d $Q(x) - P(x)$

$x^2 - 8x - 1$

e $P(x) \times Q(x)$

$5x^3 - 13x^2 - x + 2$

f $Q(x) \times P(x)$

$5x^3 - 13x^2 - x + 2$

3 When multiplying two polynomials, $P(x)$ and $Q(x)$, the degree of the resulting product $P(x)Q(x)$ will be the sum of the degrees of $P(x)$ and $Q(x)$. Explain why this is true.

The index law of multiplication says we add powers when multiplying the same base.

4 Give the degree of the polynomial $P(x) \times Q(x)$ when

a $P(x)$ is quadratic and $Q(x)$ is linear

3

b $P(x)$ is quadratic and $Q(x)$ is cubic

5

c $P(x)$ is cubic and $Q(x)$ is quartic

7

d $P(x)$ is of degree 7 and $Q(x)$ is of degree 5

12

5 If $P(x) = x^2 - 2x + 1$ and $Q(x) = x^3 + x - 1$, what is the degree of:

a $P(x) \times Q(x)$

5

b $\left(P(x)\right)^2$

4

c $\left(Q(x)\right)^2$

6

6 If $P(x) = 3x - 2$ and $Q(x) = x^2 - 2x + 1$, find expressions for:

a $P(x) + Q(x)$

$x^2 + x - 1$

b $P(x) - Q(x)$

$-x^2 + 5x - 3$

c $Q(x) - P(x)$

$x^2 - 5x + 3$

d $2Q(x) - 3P(x)$

$2x^2 - 13x + 8$

e $P(x)Q(x)$

$3x^3 - 8x^2 + 7x - 2$

f $P(x)Q(x) + 2Q(x)$

$3x^3 - 6x^2 + 3x$

7 $P(x) = ax^3 + 4$. If $P(2) = 28$,
find the value of the leading coefficient a and hence find the polynomial.

$P(x) = 3x^3 + 4$

8 $P(x) = ax^4 + bx^3$. If $P(1) = 3$ and $P(-2) = 96$,
use simultaneous equations to find a and b and hence find the polynomial.

$P(x) = 5x^4 - 2x^3$

9 A monomial is an expression with a single term. Complete these divisions, writing the quotient as a sum, or difference, of separate terms.

a $x^7 \div x^4$ x^3	**b** $\frac{10x^{10}}{2x^2}$ $5x^8$	**c** $\frac{x^3}{15} \div \frac{x}{3}$ $\frac{x^2}{5}$
d $(x^5 + x) \div x^2$ $x^3 + x^{-1}$	**e** $(24x^3 + 16x^2) \div 8x^2$ $3x + 2$	**f** $\frac{10x^6 - 12x^4 + 2x^2}{2x^3}$ $5x^3 - 6x + x^{-1}$

When is the quotient a polynomial?

A quotient is a polynomial when all the terms have non-negative integer powers.
This happens when all terms in the numerator have powers greater than or equal to the power in the denominator.

10 Consider $P(x) = ax^4 + bx^3 + cx^2 + dx + e$.

a Show that if $P(x)$ is an even function, then $b = d = 0$.

b Using your result from part a, explain why for an even polynomial, if $x = \alpha$ is a root, then $x = -\alpha$ is also root.

Even polynomial means $P(\alpha) = P(-\alpha)$.
From part a, every even polynomial only has even powers,
that means if α satisfies the equation $P(\alpha) = 0$, then $P(-\alpha)$ would also satisfy the equation,
as raising a negative to an even power is the same as raising the positive to an even power.

11 Consider $Q(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$.

a Show that if $Q(x)$ is an odd function, then $b = d = f = 0$.

b Using your result from part a, explain why every odd polynomial must pass through the origin.

From part a, every odd polynomial has terms with odd power.
That means we can factorise x out, hence $x = 0$ is a root, i.e. it passes through the origin.