Student Name

Mathematics Stage 5 Paths

PROPERTIES OF GEOMETRICAL FIGURES

Book 1 Construct formal proofs involving congruent and similar triangles

Version: 260207

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CONTENTS

OVERVIEW

SYLLABUS CONTENT

MA5-GEP-P-02 constructs proofs involving congruent triangles and similar triangles and proves properties of plane shapes

Construct formal proofs involving congruent and similar triangles

- Construct formal proofs of the congruence of triangles, preserving the matching order of vertices and drawing relevant conclusions
- Construct formal proofs of the similarity of triangles, preserving the matching order of vertices and drawing relevant conclusions

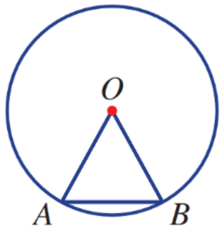
REVIEW OF PRIOR KNOWLEDGE

1 The diagram shows a circle with centre O.

a Which two sides of ΔOAB are equal? Why?

……………………………………………………

……………………………………………………



b Which two angles are equal? Why?

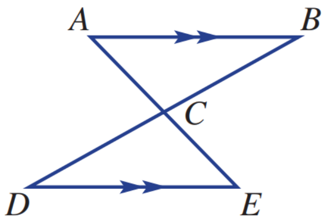
……………………………………………………

……………………………………………………

2 Consider the diagram shown below.

a Which angle is vertically opposite to ∠ACB?

……………………………………………………



b Which angle is alternate to ∠CDE?

……………………………………………………

c Which angle is alternate to ∠BAC?

……………………………………………………

d Explain why this is not enough information to conclude that ΔABC ≡ ΔDCE.

……………………………………………………………………………………………………

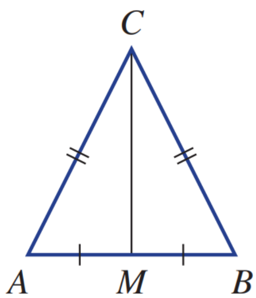
e What is one piece of information you can add to the diagram such that ΔABC ≡ ΔDCE? State the congruence test you are using.

……………………………………………………………………………………………………

3 For this isosceles triangle, CM is common to ΔAMC and ΔBMC.

a Which congruence test would be used to prove that ΔAMC ≡ ΔBMC?

……………………………………………………

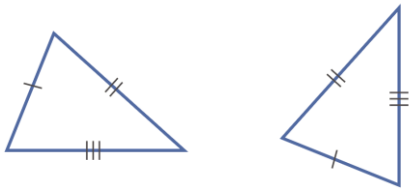
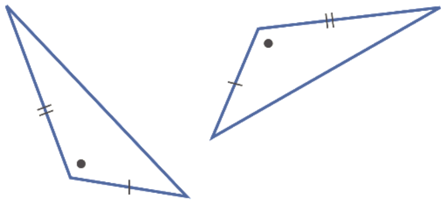
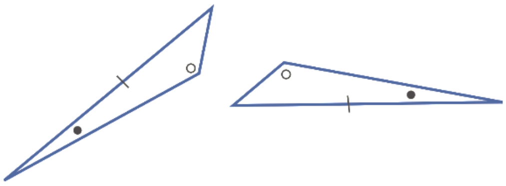
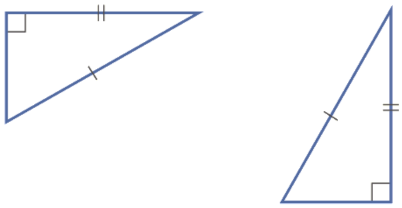


b Which angle in ΔBMC corresponds to ∠AMC in ΔAMC?

……………………………………………………

FORMAL PROOFS OF CONGRUENCE

- If two triangles satisfy one of these four tests, then the two triangles are congruent.

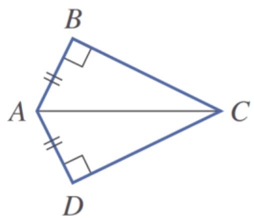
SSS	SAS
 Three pairs of corresponding sides.	 Two pairs of corresponding sides and the included angle equal.
AAS	**RHS**
 Two pairs of corresponding angles and one pair of corresponding sides equal.	 For a right-angled triangle, the hypotenuse and another corresponding side equal.

FORMAL PROOFS

- A proof starts with given facts, shows each step with reasons, and has no gaps in logic.
  1. Decide on the congruence proof (SSS, SAS, AAS, or RHS)
  2. Write a formal proof, giving reasons.

Example Prove two triangles are congruent.

Prove that $\Delta ABC \equiv \Delta ADC$



In ΔABC and ΔACD:

R: $\angle ABC = \angle ADC = 90°$ (given)

H: AC is common

S: $AB = AD$ (given)

$\therefore \Delta ABC \equiv \Delta ADC$ (RHS)

what does the reason "given" mean?

The information is given by the ………………

What does "AC is common" mean?

AC is a ……………… side in both triangles.

Complete the proofs.

a Prove that $\Delta ABC \equiv \Delta EDC$

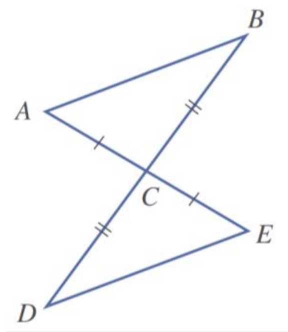
In ΔABC and ΔEDC:

S: $AC = CE$ (……………)

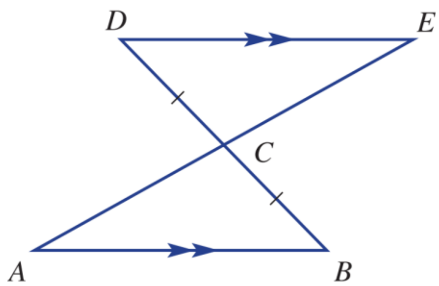
A: $\angle ACB = \angle DCE$ (…………… ………………)

S: $DC = CB$ (……………)

$\therefore \Delta ABC \equiv \Delta EDC$ (……………)



b Prove $\Delta ABC \equiv \Delta EDC$



In Δ……… and Δ………

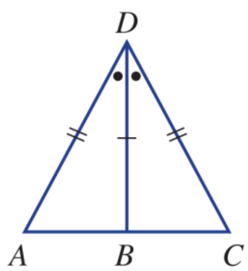
A: ∠……. = ∠ ……. (vertically opposite)

A: $\angle ABC = \angle EDC$ …………………………

S: $BC = DC$ …………………………

$\therefore \Delta ABC \equiv$ ………… (…………)

c Prove $\Delta ADB \equiv \Delta CDB$



In ……… and ………

S: ………………… …………………………

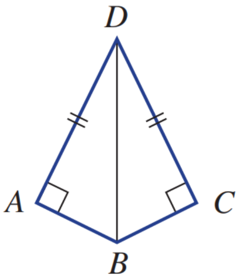
A: ………………… …………………………

S: ………………… …………………………

∴ ……… ≡ ……… (…………)

1 Prove that each pair of triangles is congruent.
List your reasons and give the abbreviated congruence test.

a



In Δ ……… and Δ ………:

ΔABD and ΔCBD

R: ………………… (…………………………)

$\angle DAB = \angle DCB$ (given)

H: …………………

DB is common

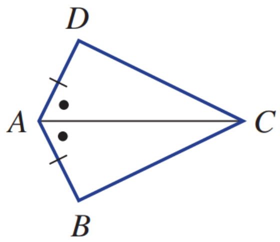
S: ………………… (…………………………)

$AD = CD$ (given)

∴ Δ……… ≡ Δ……… (…………)

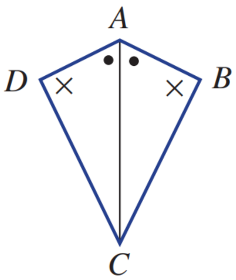
$\Delta ABD \equiv \Delta CBD$ (RHS)

b



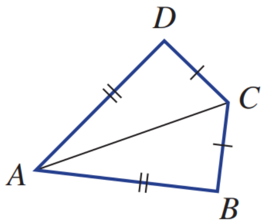
In ΔADC and ΔABC:
S: AC is common
A: ∠DAC = ∠CAB (given)
S: AD = AB (given)
∴ Δ ADC ≡ Δ ABC (SAS)

c



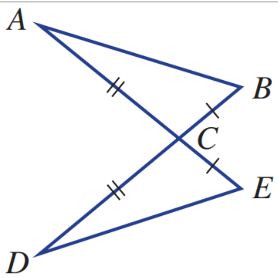
In ΔADC and ΔABC
A: ∠ADC = ∠ABC (given)
A: ∠DAC = ∠BAC (given)
S: AC is common
∴ Δ ADC ≡ Δ ABC (AAS)

d



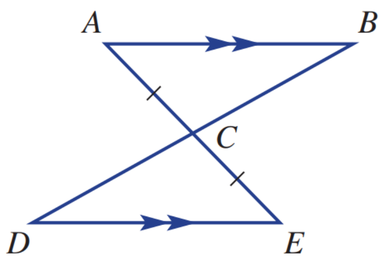
In ΔADC and ΔABC
S: AD = AB (given)
S: AC is common
S: DC = CB (given)
∴ Δ ADC ≡ Δ ABC (SSS)

e



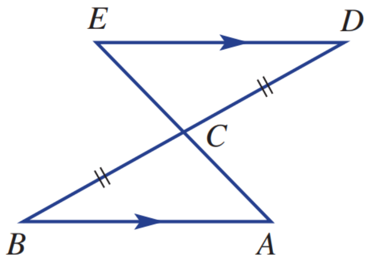
In ΔABC and ΔDEC
S: AC = CD (given)
A: ∠ACB = ∠DCE (vertically opposite)
S: CE = CB (given)
∴ Δ ABC ≡ Δ DEC (SAS)

f



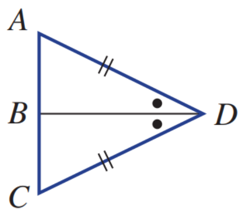
In ΔABC and ΔEDC
A: ∠DCE = ∠ACB (vertically opposite)
A: ∠BAC = ∠DEC (alternate angles on parallel lines)
S: AC = EC (given)
∴ Δ ABC ≡ Δ EDC

g



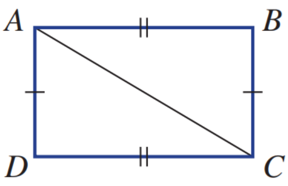
In ΔCAB and ΔCED
A: ∠ECD = ∠ACB (vertically opposite)
A: ∠DEC = ∠BAC (alternate angles on parallel lines)
S: BC = DC (given)
∴ ΔCAB ≡ ΔCED (AAS)

h



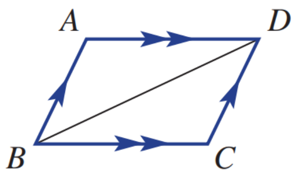
In ΔADB and ΔCDB
S: AD = CD (given)
A: ∠ADB = ∠CDB (given)
S: BD is common
∴ ΔADB ≡ ΔCDB (SAS)

i



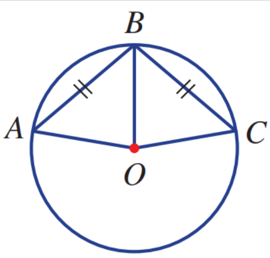
In ΔACD and ΔCAB
S: AD = CB (given)
S: AC is common
S: AB = CD (given)
∴ ΔACD ≡ ΔCAB (SSS)

j



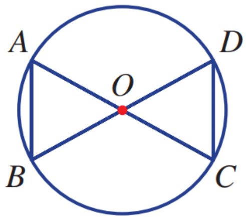
In ΔADB and ΔCBD
A: ∠ADB = ∠CBD (alternate angles on parallel lines)
A: ∠ABD = ∠CDB (alternate angles on parallel lines)
S: BD is common
∴ ΔADB ≡ ΔCBD (AAS)

k



In ΔOAB and ΔOCB
S: AB = CB (given)
S: OB is common
S: OA = OC (radii of circle)
∴ ΔOAB ≡ ΔOCB (SSS)

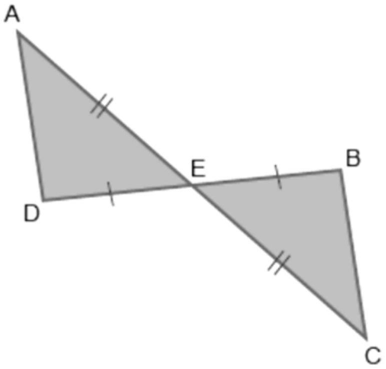
l



In ΔOAB and ΔODC
S: OA = OD (radii of circle)
A: ∠AOB = ∠DOC (vertically opposite)
S: OB = OC (radii of circle)
∴ ΔOAB ≡ ΔODC (SAS)

2

a Prove $\Delta AED \equiv \Delta \mathrm{CEB}$



In ΔAED and ΔCEB

S: AE = CE (given)

A: ∠AED = ∠CEB (vertically opposite)

S: DE = BE (given)

∴ Δ AED ≡ Δ CEB (SAS)

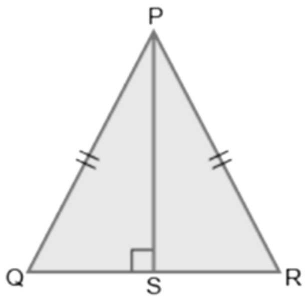
b Hence, prove that $\angle A = \angle C$:

$\angle A = \angle C$ (corresponding angles in …………………… triangles are equal)

congruent

3

a Prove $\Delta PSQ = \Delta PSR$



In ΔPSQ and ΔPSR

R: ∠PSQ = ∠PSR = 90° (angles on a straight line)

H: PQ = PR (given)

S: PS is common

∴ Δ PSQ ≡ Δ PSR (RHS)

b Prove $QS = SR$

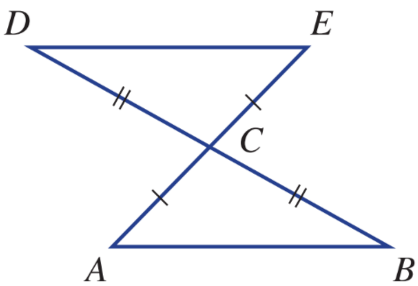
QS = SR (corresponding sides on congruent triangles equal)

c Hence, explain why PS bisects QR (bisect means cut exactly in half)

Since QS = SR, PS bisects QR

4

a Prove ΔABC ≡ ΔEDC.



In ΔABC and ΔEDC

S: DC = BC (given)

A: ∠DCE = ∠BCA (vertically opposite)

S: CE = CA (given)

∴ Δ ABC ≡ Δ EDC (SAS)

b Hence, prove AB ∥ DE.

$\angle DEC = \angle CAB$ (corresponding angles in ……………… triangles)

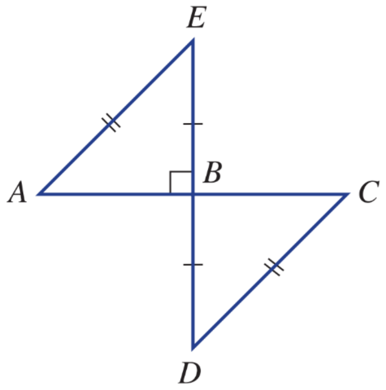
$AB||DE$ (………………… angles on parallel lines are ………..)

congruent

alternate, equal

5

a Prove ΔABE ≡ ΔCBD.



AE = CD (given)

BE = BD (given)

∠ABE = ∠CBD (vertically opposite with ∠ABE given 90°)

∴ ΔABE ≡ ΔCBD (RHS)

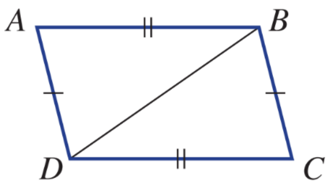
b Hence, prove AE ∥ DC.

∠EAB = ∠DCB (corresponding angles in congruent triangles)

∴ AE || CD (alternate angles equal)

6

a Prove ΔABD ≡ ΔCDB.



DB is common

AB = CD (given)

AD = CB (given)

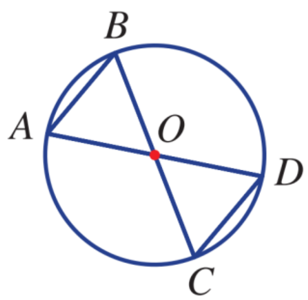
∴ ΔABD ≡ ΔCDB (SSS)

b Hence, prove AD ∥ BC.

∠ADB = ∠CBD (corresponding angles in congruent triangles)

∴ AD || BC (alternate angles equal)

c Prove that ΔAOB ≡ ΔDOC. (O is the centre of the circle)



OB = OC (radii)

OA = OD (radii)

∠AOB = ∠DOC (vertically opposite)

∴ ΔAOB ≡ ΔDOC (SAS)

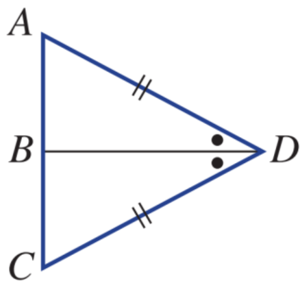
d Hence, prove that AB ∥ CD.

∠ABO = ∠DCO (corresponding angles in congruent triangles)

∴ AB || CD (alternate angles equal)

7

a Prove that ΔABD ≡ ΔCBD.



BD is common

AD = CD (given)

∠ADB = ∠CDB (given)

∴ ΔABD ≡ ΔCBD (SAS)

b Hence, prove that AC is perpendicular to BD. (AC ⊥ BD)

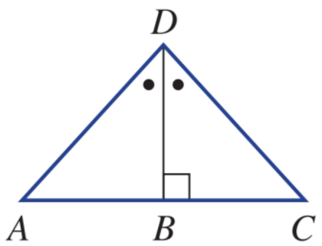
∠ABD = ∠CBD (corresponding angles in congruent triangles)

and ∠ABD + ∠CBD = 180° (straight line)

∴ ∠ABD = ∠CBD = 90° and AC is perpendicular to BD

8

a Prove that ΔABD ≡ ΔCBD.



DB is common

∠ABD = ∠CBD (given 90°)

∠ADB = ∠CDB (given)

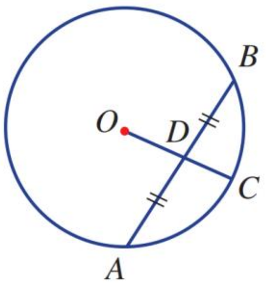
∴ ΔABD ≡ ΔCBD (AAS)

b Hence, prove that ΔACD is isosceles.

AD = CD (corresponding side in congruent triangles)

∴ ΔACD is isosceles (2 equal sides)

9 Use congruence to explain why OC is perpendicular to AB in this diagram.



Consider ΔOAD and ΔOBD

OD is common

OA = OB (radii)

AD = BD (given)

∴ ΔOAD ≡ ΔOBD (SSS)

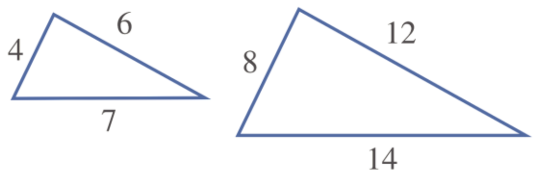
∠ODA = ∠ODB = 90°

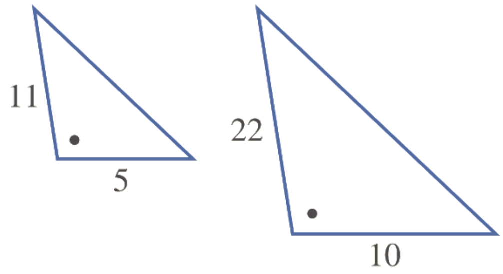
(corresponding angles in congruent triangles are equal and supplementary to a straight line)

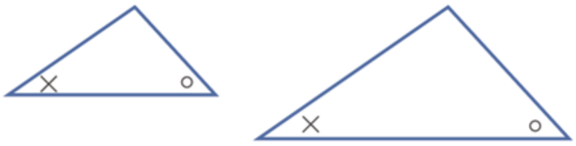
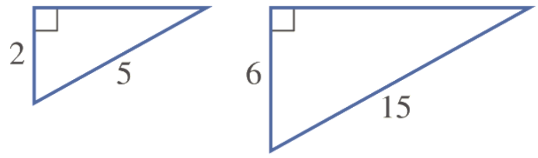
∴ OC ⊥ AB

FORMAL PROOFS OF SIMILARITY

- There are four ‘tests’ that can be used to decide if two triangles are similar.
- Unlike the congruence tests, you must write the full reasoning every time.

<table>
<tr><th>SSS</th><th>SAS</th></tr>
<tr><td>

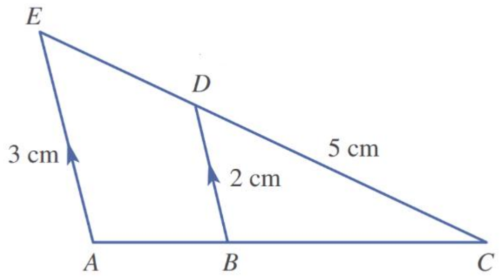
Three pairs of corresponding sides in the same ratio.</td><td>

Two pairs of corresponding sides in the same ratio and the included angle equal.</td></tr>
<tr><th>AAA</th><th>RHS</th></tr>
<tr><td>
At least two pairs of corresponding angles. (If there are two equal pairs, the third must be equal)</td><td>

For a right-angled triangle, the hypotenuses and another pair of corresponding sides in the same ratio.</td></tr>
</table>

Example Prove two triangles are similar.

Prove that $\Delta ACE \sim \Delta BCD$



In ΔABC and ΔADC:

A: $\angle ECA = \angle DCB$ (common angle)

A: $\angle EAC = \angle DBC$ (corresponding angles $EA||DB$)

A: $\angle AEC = \angle BDC$ (corresponding angles $EA||DB$)

$\therefore\ \Delta ACE \sim \Delta BCD$ (equiangular)

What does the reason "common angle" mean?

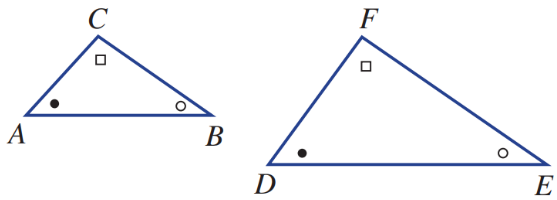
The angle is a ……………… angle in both triangles.

How do we know that $EA \parallel DB$?

It is shown by ………………… in the diagram.

Complete the proofs.

a Prove that $\Delta ACB \sim \Delta DFE$



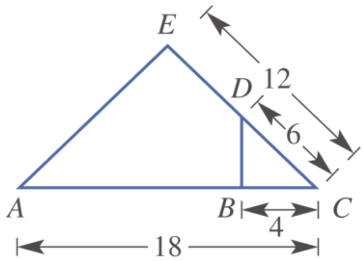
In ……… and ………

A: ………………… (…………………………)

A: ………………… (…………………………)

$\therefore$ ……… ~ ……… (……………………)

b



In ……… and ………

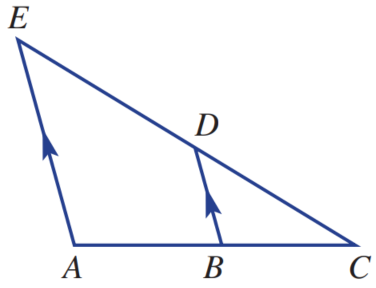
S: $\frac{AC}{DC} = - =$ …..

A: …………………

S: …………………

$\therefore$ ……… ~ ……… (…………………………
…………………………
…………………………
…………………………)

1 In this diagram:



a use three letters to name the common angle.

$\angle ECA$

b why is ∠EAC = ∠DBC?

Corresponding angles on parallel lines

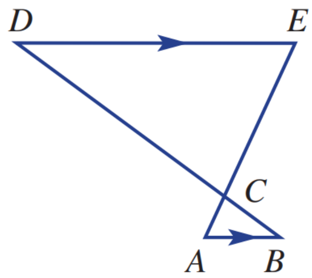
c explain why we have enough information to conclude that the triangles are similar.

At least two angles are equal, so the triangle is equiangular.

d complete the similarity statement: ΔBCD ~ Δ………

ACE

2 In this diagram:



a name the pair of vertically opposite angles

$\angle DCE = \angle BCA$

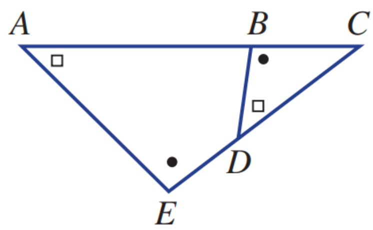
b name the two pairs of equal alternate angles

$\angle EDC = \angle ABC$

c complete the similarity statement: ΔABC ~ Δ…………

EDC

3 In this diagram name the:



a common angle for the two triangles

$\angle ECA$

b side that corresponds to side:

i DC …………… **ii** AE ……………

AC DC

c Write a similarity statement. ………………………

$\Delta ACE \sim \Delta DCB$

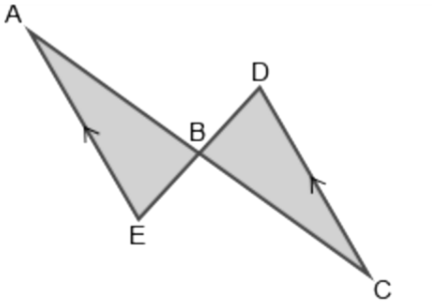
4 Prove these triangles are similar.

a In ΔABE and Δ______,

∠ABE = ∠______ (________________________________)

∠BAE = ∠______ (________________________________)

∴ Δ______ ||| Δ______ (________________________________)



In ΔABE and ΔCBD

$\angle ABE = \angle CBD$ (vertically opposite)

$\angle BAE = \angle BCD$ (alternate angles on parallel lines)

$\therefore \Delta ABE \parallel\!| \Delta CBD$ (equiangular)

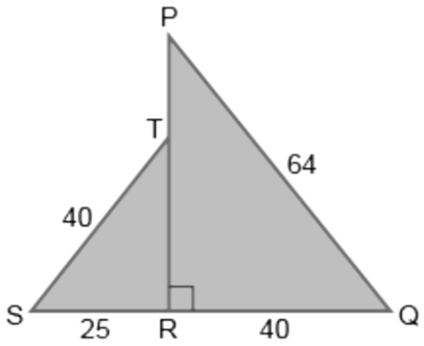
b In ΔPRQ and Δ______,

∠PRQ = ∠______ (= 90º, given)

$\frac{PQ}{TS} = \frac{\square}{\square} = \frac{\square}{\square}$ (lengths given)

$\frac{RQ}{\square} = \frac{\square}{\square} = \frac{\square}{\square}$ (lengths given)

∴ ΔPRQ ||| Δ______ (________________________________)



In ΔPRQ and ΔTRS

$\angle PRQ = \angle TRS$ (= 90°, given)

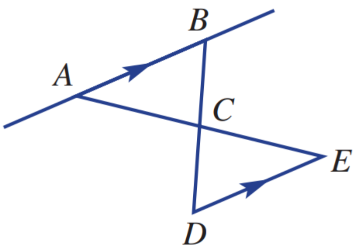
$\frac{PQ}{TS} = \frac{64}{40} = \frac{8}{5}$

$\frac{RQ}{RS} = \frac{40}{25} = \frac{8}{5}$

$\therefore \Delta PRQ \parallel\!| \Delta TRS$ (the hypotenuses and another pair of corresponding sides in the same ratio.)

5 Prove that each pair of triangles is similar.

a



∠BAC = ∠DEC (alternate angles, AB || DE)

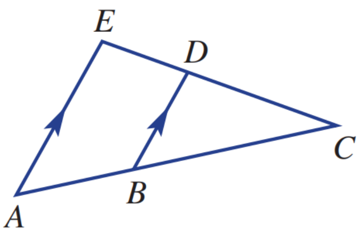
∠ABC = ∠EDC (alternate angles, AB || DE)

∠ACB = ∠ECD (vertically opposite)

∴ ΔACB ||| ΔECD (equiangular)

Note: only two angles are needed.

b



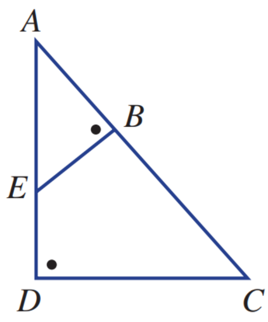
∠C is common

∠CDB = ∠CEA (corresponding angles, AE || BD)

∠CBD = ∠CAE (corresponding angles, AE || BD)

∴ ΔCBD ||| ΔCAE (equiangular)

c

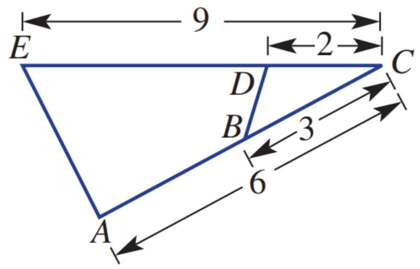


∠A is common

∠ABE = ∠ADC (given)

∴ △ABE ||| △ADC (equiangular)

d



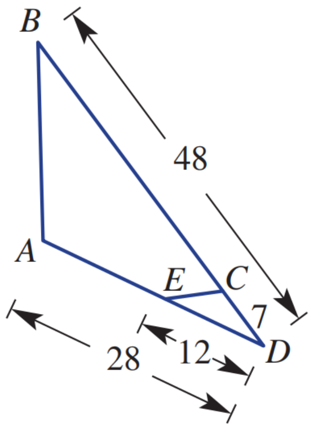
∠C is common

$$\frac{CA}{CD} = \frac{6}{2} = 3$$

$$\frac{CE}{CB} = \frac{9}{3} = 3$$

∴ △CDB ||| △CAE (Two pairs of corresponding sides in the same ratio and the included angle equal.)

e



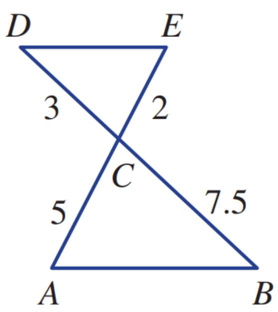
∠D is common

$$\frac{AD}{CD} = \frac{28}{7} = 4$$

$$\frac{DB}{DE} = \frac{48}{12} = 4$$

∴ △ABD ||| △CED (Two pairs of corresponding sides in the same ratio and the included angle equal.)

f



∠DCE = ∠BCA (vertically opposite)

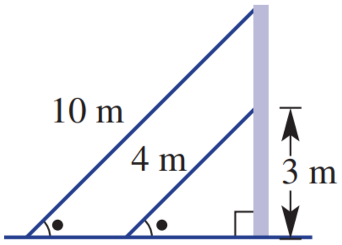
$$\frac{EC}{AC} = \frac{2}{5}$$

$$\frac{DC}{BC} = \frac{3}{7.5} = \frac{2}{5}$$

∴ △DCE ||| △BCA (Two pairs of corresponding sides in the same ratio and the included angle equal.)

6 Two cables support a steel pole at the same angle as shown.
The two cables are 4 m and 10 m in length while the shorter cable reaches 3 m up the pole.

a Give a reason why the two triangles are similar (just the reason, not the formal proof)



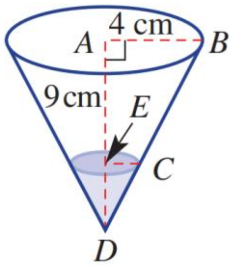
Equiangular

b Find the height of the pole.

40 m

7 A right cone with radius 4 cm has a total height of 9 cm. It contains an amount of water.
It is held such that the water surface EC is parallel to the top surface of the cone AB.

a Prove △EDC ||| △ADB.



$\angle BDA$ is common

$\angle BAD = \angle ECD$ (corresponding angles on || lines)

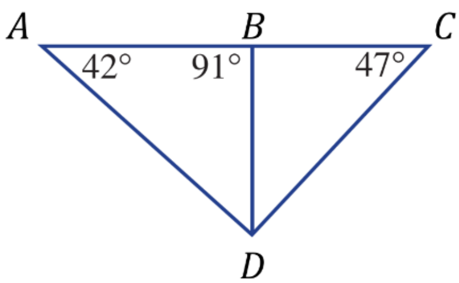
∴△EDC ||| △ADB (equiangular)

b If the depth of water in the cone is 3 cm, find the radius of the water surface in the cone.

$$\frac{3}{9} = \frac{EC}{4}$$

$$\therefore \frac{4}{3}\ \text{cm}$$

8 In the diagram, prove that there is a pair of similar triangles.



$\angle BDA = 180 - 42 - 91 = 47°$ (angle sum of ΔABD)

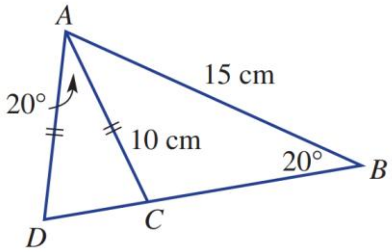
$\angle BAD$ is common

$\angle BDA = \angle DCB$

$\therefore \Delta BDA|||\Delta DCB$ (equiangular)

9 In this diagram ΔADC is isosceles.

a There are two triangles that are similar. Identify them and complete a proof.
You may need to find another angle first.



$\angle ADC = \angle ACD = 80^\circ$ (base angles of isosceles $\triangle ADC$)

$\angle ACB = 100^\circ$ (straight angle)

$\angle CAB = 60^\circ$ (angle sum of $\triangle ACB$)

Now $\angle DAB = 80^\circ$

$\angle DAC = \angle DBA$ (given 20°)

$\angle D$ is common

$\angle ACD = \angle BAD$ (both 80°)

$\therefore \triangle ACD \,|||\, \triangle BAD$ (equiangular)

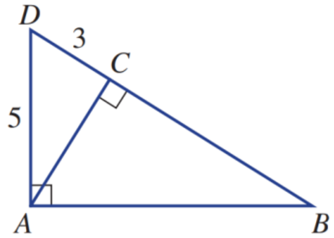
b Find the lengths DC and CB, expressing your answers as fractions.

$DC = \frac{20}{3}, CB = \frac{25}{3}$

10 In this diagram AC is perpendicular to BD and ΔABD is right angled.

a Prove that ΔABD is similar to:

i ΔCBA



$\angle B$ is common

$\angle DAB = \angle ACB$ (given 90°)

$\therefore \triangle ABD \,|||\, \triangle CBA$ (equiangular)

ii CAD

$\angle D$ is common

$\angle DCA = \angle DAB$ (given 90°)

$\therefore \triangle ABD \,|||\, \triangle CAD$ (equiangular)

b Find the lengths:

i BD

$\frac{25}{3}$

ii AC

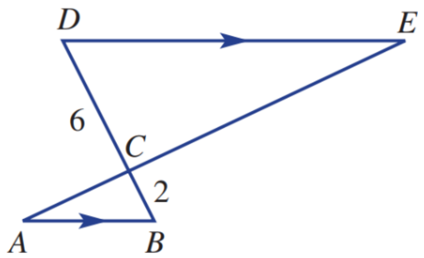
4

iii AB

$\frac{20}{3}$

11 Complete these proofs.

a Prove $AE = 4AC$.



∠ACB = ∠ECD (vertically opposite)

∠CAB = ∠CED (alternate angles, DE || BA)

∴ ∆ABC ||| ∆EDC (AAA)

∴ $\frac{DC}{BC} = \frac{EC}{AC}$ (ratio of corresponding sides in similar triangles)

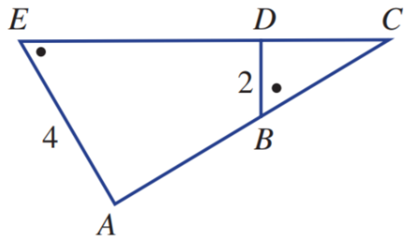
$\frac{6}{2} = \frac{EC}{AC}$

∴ 3AC = CE

as AC + CE = AE

AE = 4AC

b Prove $BC = \frac{1}{2}CE$.



∠C is common

∠DBC = ∠AEC (given)

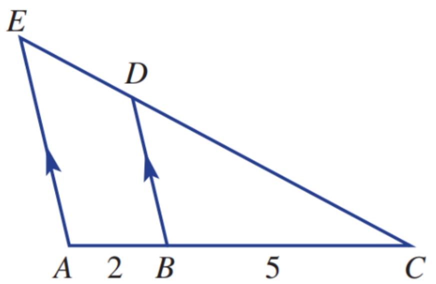
∴ ∆CBD ||| ∆CEA (AAA)

∴ $\frac{DB}{AE} = \frac{2}{4} = \frac{BC}{CE}$ (ratio of corresponding sides in similar triangles)

∴ 4BC = 2CE

$BC = \frac{1}{2}CE$

c Prove $CE = \frac{7}{5}CD$.



∠C is common

∠CBD = ∠CAE (corresponding angles, BD || AE)

∴ ∆CBD ||| ∆CAE (AAA)

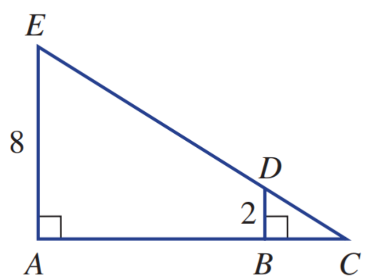
∴ $\frac{CB}{CA} = \frac{CD}{CE}$ (ratio of corresponding sides in similar triangles)

$\therefore \frac{5}{7} = \frac{CD}{CE}$

∴ 5CE = 7CD

$\therefore CE = \frac{7}{5}CD$

d Prove $AB = 3BC$.



∠C is common

∠CBD = ∠CAE (given 90°)

∴ ∆CBD ||| ∆CAE (AAA)

∴ $\frac{BD}{AE} = \frac{CB}{CA}$ (ratio of corresponding sides in similar triangles)

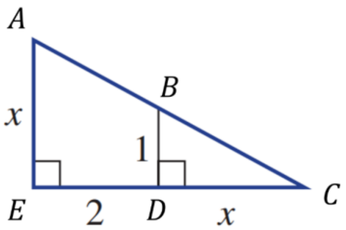
$\frac{2}{3} = \frac{CB}{CA}$

$\frac{1}{4} = \frac{CB}{CB + AB}$

CB + AB = 4CB

∴ AB = 3CB

12 Find the value of x.



$\angle ACE$ is common

$\angle AEC = \angle DBC$ (given)

$\therefore \Delta ACE \sim \Delta BCD$ (equiangular)

$$\frac{1}{x} = \frac{x}{x+2}$$

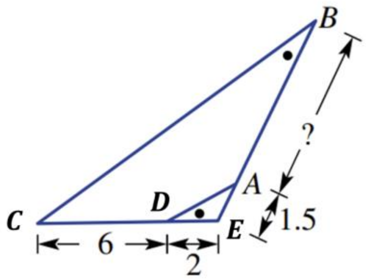
$$x + 2 = x^2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2$$

13 Find the length of AB.



$\angle CBE = \angle ADE$ (given)

$\angle E$ is common

$\therefore \Delta CBE \sim \Delta ADE$ (equiangular)

$$\frac{DE}{BE} = \frac{AE}{CE}$$

$$\frac{2}{AB + 1.5} = \frac{1.5}{8}$$

$$16 = 1.5(AB) + 2.25$$

$$1.5(AB) = 13.75$$

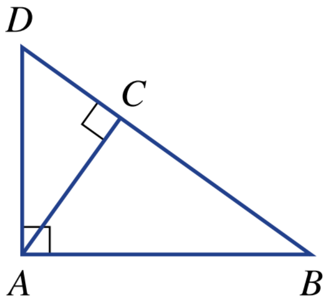
$$AB = \frac{55}{6}$$

14 In this figure, ΔABD, ΔCBA and ΔCAD are right angled.

a Prove ΔABD ||| ΔCBA. Hence, prove $AB^2 = CB \times BD$.

b Prove ΔABD ||| ΔCAD. Hence, prove $AD^2 = CD \times BD$.

c Hence, prove Pythagoras' theorem $AB^2 + AD^2 = BD^2$.



$\angle BAD = \angle BCA = 90^\circ$

$\angle ABD = \angle CBA$ (common)

So ΔABD ||| ΔCBA (AAA).

Therefore, AB/CB = BD/AB.

$AB^2 = CB \times BD$

$\angle BAD = \angle ACD = 90^\circ$

$\angle ADB = \angle CDA$ (common)

So ΔABD ||| ΔCAD (AAA)

Therefore, AD/CD = BD/AD.

$AD^2 = CD \times BD$

Adding the two equations:

$AB^2 + AD^2 = CB \times BD + CD \times BD$

$= BD(CB + CD)$

$= BD \times BD$

$= BD^2$