Student Name

Mathematics Stage 5 Paths

POLYNOMIALS

Book 3 Apply the factor and remainder theorems to solve problems

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CONTENTS

OVERVIEW

SYLLABUS CONTENT

MA5-POL-P-01 defines, operates with and graphs polynomials and applies the factor and remainder theorems to solve problems

Apply the factor and remainder theorems to solve problems

- Verify the remainder theorem and use it to find factors of polynomials and solve related problems
- Develop and apply the factor theorem to factorise particular polynomials completely and solve related problems
- Apply the factor theorem and division to find the zeroes of a polynomial $P(x)$ and solve $P(x) = 0$ (degree ≤ 4)
- State the maximum number of zeroes a polynomial of degree n can have

REVIEW OF PRIOR KNOWLEDGE

1 Solve these equations by using the null factor law.

a $(x-1)(x+3)=0$	**b** $(x+2)(x-2)=0$
c $(2x+1)(x-4)=0$	**d** $(x+3)(x-2)(x-1)=0$
e $(x+2)(x+7)(x-1)=0$	**f** $(x-4)(x+4)(x-3)=0$
g $(2x+1)(x-3)(3x+2)=0$	**h** $(4x-1)(5x-2)(7x+2)=0$

2 Write down in factored form:

a A monic cubic polynomial with zeroes $-1, 3,$ and 4. ....

b a monic quartic polynomial with zeroes 0, −2, 3 and 1 ....

REMAINDER THEOREM

Remainder Theorem
For any linear divisor $(x-a)$, The remainder of $\frac{P(x)}{(x-a)}$ is $P(a)$.

Investigation Remainder Theorem.

Divide $P(x) = x^3 - x^2 + 2x - 3$ by $(x-2)$

Evaluate $P(2)$ and show that the result is the remainder.

Example Use remainder theorem to find remainder after division.

Find the remainder when $3x^4 - 4x^3 + 4x - 8$ is divided by $x - 2$

$P(x) = 3x^4 - 4x^3 + 4x - 8$

$P(2) = 3(2)^4 - 4(2)^3 + 4(2) - 8$

$= 16$

Find the remainder when $P(x) = x^3 - 5x^2 - x + 4$ is divided by:

a $x - 2$

$P(2) =$

-10

b $x + 1$

-1

c $x - \frac{1}{2}$

$\frac{19}{8}$

d $x + \frac{2}{3}$

$\frac{58}{27}$

1 Without division, find the remainder when $P(x) = x^3 - x^2 + 2x + 1$ is divided by:

a $x - 1$ $P(1) =$ 3	**b** $x - 3$ 25	**c** $x + 2$ -15
d $x + 1$ -3	**e** $x - 5$ 111	**f** $x + 3$ -41

2 Without division, find the remainder when $P(x) = x^4 - x^3 + 3x^2$ is divided by:

a $x - 1$ 3	**b** $x - 2$ 20	**c** $x + 2$ 36

3 For what value of k will $(x^4 - 2x^3 + x^2 - x + k) \div (x + 2)$ have zero remainder?

Let $P(-2) = 0$

-38

4 Find the value of k in these polynomials.

a When $P(x) = x^3 + 2x^2 + kx - 4$ is divided by $(x - 1)$, the remainder is 4.

5

b When $P(x) = 2x^3 + kx^2 + 3x - 4$ is divided by $(x + 2)$, the remainder is −6.

5

c When $P(x) = kx^3 + 7x^2 - x - 4$ is divided by $(x - 2)$, the remainder is −2.

-3

5 For non-monic divisors: $P\left(\frac{b}{a}\right)$ is the remainder when $P(x)$ is divided by $(ax - b)$.
Without division, find the remainder when $P(x) = x^3 + 2x^2 - 4x + 5$ is divided by:

a $2x - 1$	**b** $2x + 3$	**c** $3x - 2$
$P\left(\frac{1}{2}\right) =$ $\frac{29}{8}$	$\frac{97}{8}$	$\frac{95}{27}$

6 The remainder when $P(x) = (mx^4 - 16x^3 + 6x)$ is divided by $(2x + 1)$ is $-\frac{1}{4}$.
Find the value of m.

12

7 The remainder when $x^2 + 6x + 6$ is divided by $(x - c)$ is 1.
Find the values of c and hence the divisors.

$c = -5 \text{ or } -1$

Divisors are $(c + 5) \text{ or } (x + 1)$

8 The remainder when $x^2 - x - 50$ is divided by $(x - c)$ is -8.
Find the values of c and hence the divisors.

$c = 7 \text{ or } c = -6$

Divisors are $(x - 7), \text{or } (x + 6)$

9 Find the divisors of $10x^2 + 3x - 25$ that leave a remainder of 2.

$\left(x - \frac{3}{2}\right) \text{ or } \left(x + \frac{9}{5}\right)$

10 The remainder when $x^4 - 21x^2 + 70$ is divided by $(x - c)$ is -10.
Find the values of c and hence the divisors. There are four answers for c.

Hint: this a quadratic in disguise.

$(x-4)$ or $(x+4)$ or $(x-\sqrt{5})$ or $(x+\sqrt{5})$

11 When $P(x) = 2x^3 - x^2 + ax + b$ is divided by $(x - 1)$ the remainder is 16, and when it is divided by $(x + 2)$ the remainder is −17. Find a and b using simultaneous equations

$a = 4$ and $b = 11$

12 $P(x) = 2x^3 - ax^2 - bx - 1$. The remainder is -10 when $P(x)$ is divided by $(x + 1)$ and -37 when $P(x)$ is divided by $x + 2$. Find a and b.

$a = 3, b = -4$

13 Prove that, in general, if $P(x) \div (x - a) = Q(x) + \frac{r}{x-a}$, then $P(a) = r$.

(i.e. prove the remainder theorem)

Multiply both sides by $(x-a)$

$P(x) = (x-a)Q(x) + r$

$P(a) = (a-a)Q(a) + r$

$\therefore P(a) = r$

FACTOR THEOREM

Factor Theorem
$(x - a)$ is a factor of $P(x)$ if $P(a) = 0$. a is called a **zero** of the polynomial. In general, $(ax - b)$ is a factor of $P(x)$ if $P\left(\frac{b}{a}\right) = 0$

Fundamental Theorem of Algebra
A polynomial of degree n has at most n factors.

Investigation Factor Theorem.

Divide $P(x) = x^3 - 3x^2 - 3x + 10$ by $(x - 2)$

Evaluate $P(2)$ and show that it is zero.

Explain how factor theorem is related to remainder theorem:

The remainder theorem states that the remainder of $\frac{P(x)}{(x-a)}$ *is ....*

If $P(a) = 0$*, that means there is .... remainder, hence* $(x - a)$ *divides evenly into* $P(x)$

Example Use factor theorem to find factors of a polynomial.

Show that $(x - 3)$ is a factor of $P(x) = x^3 - 2x^2 + x - 12$ and $(x + 1)$ is not.

$P(3) = (3)^3 - 2(3)^2 + (3) - 12$

$= 0$ so $x - 3$ is a factor.

$P(-1) = (-1)^3 - 2(-1)^2 + (-1) - 12$

$= -16$ so $x - 1$ is not a factor.

Determine whether each of the following are factors of $P(x) = x^3 + x^2 - 3x - 6$

a $x + 1$

$P(-1) \neq 0,\ (x + 1)$ not a factor

b $x - 2$

$P(2) = 0,\ (x - 2)$ is a factor

1 What is the remainder when an expression is divided by one of its factors?

0

2 Without division, find which of the following are factors of $F(x) = x^3 + 4x^2 + x - 6$.

a $x - 1$ yes	**b** $x + 1$ no	**c** $x - 2$ no
d $x + 2$ yes	**e** $x - 3$ no	**f** $x + 3$ yes

3 Without division, find which of the following are factors of $F(x) = x^4 - 2x^3 - 25x^2 + 26x + 120$.

a $x - 2$ no	**b** $x + 2$ yes	**c** $x + 3$ no
d $x - 4$ no	**e** $x + 4$ yes	**f** $x - 5$ yes

4

a Find the value of k if $(x + 2)$ is a factor of $x^3 - kx^2 - 2x - 4$

-2

b Find the value of k if $(x - 3)$ is a factor of $2x^3 + 2x^2 - kx - 3$

23

c Find m, if -2 is a zero of the function $F(x) = x^3 + mx^2 - 3x + 4$.

$-\frac{1}{2}$

d For what value of a is $3x^4 + ax^2 - 2$ divisible by $x + 1$?

-1

5 Prove that $(x^2 - 4)$ is a factor of $P(x) = x^4 + 2x^3 - 3x^2 - 8x - 4$ without performing a division.

$x^2 - 4 = (x - 2)(x + 2)$

$P(2) = 0$ and $P(-2) = 0$

$\therefore x^2 - 4$ is a factor as $(x - 2)$ and $(x + 2)$ are factors.

6 Prove that $x^n - 1$ is always divisible by $(x - 1)$.

Let $P(x) = x^n - 1$

$P(1) = 1^n - 1$

Since 1^n always equals 1,

$P(1) = 0$

7 When the polynomial $P(x)$ is divided by $(x - 1)(x + 3)$, the quotient is $Q(x)$ and the remainder is $(2x + 5)$.

a Write this information in quotient-remainder form $P(x) = Q(x)D(x) + R(x)$

$P(x) = (x - 1)(x + 3)Q(x) + (2x + 5)$

b Hence, by evaluating $P(1)$, find the remainder when $P(x)$ is divided by $(x - 1)$.

7

c What is the remainder when $P(x)$ is divided by $(x + 3)$?

-1

8 When $P(x)$ is divided by $(x^2 - 9)$, the remainder is $(2x + 5)$

a Write the above in the form of $P(x) = D(x)Q(x) + R(x)$

$P(x) = (x^2 - 9)Q(x) + (2x + 5)$

b Hence, find the remainder when $P(x)$ is divided by $(x + 3)$

-1

9 The polynomial $P(x)$ is divided by $(x-1)(x+2)$.
Suppose that the quotient is $Q(x)$ and the remainder is $R(x)$.

a Explain why $R(x)$ must be in the form $ax + b$, where a and b are constants.

The divisor has degree 2, so the remainder has degree 0 or 1.

b If $P(1) = 2$ and $P(-2) = 5$, find a and b.

$a = -1, b = 3$

10 The polynomial $P(x)$ is given by $P(x) = x^3 + ax^2 + bx - 18$.
Find a and b given that $(x+2)$ is a factor of $P(x)$,
and -24 is the remainder when $P(x)$ is divided by $x - 1$.

$P(-2) = 0$

$P(1) = -24$

$a = 2$ and $b = -9$

11 The polynomial $P(x)$ is divided by $(x+4)(x-3)$.
Find the remainder if $P(-4) = 11$ and $P(3) = -3$.

$P(x) = (x+4)(x-3)Q(x) + ax + b$

$P(-4) = 11$

$P(3) = -3$

Solve simultaneously, $a = -2, b = 3$

$3 - 2x$

12 Find the values of m and n if $(x-1)$ and $(x-2)$ are factors of $P(x) = x^4 + mx + n$

$m = -15, n = 14$

13 When a polynomial $P(x)$ is divided by $(x-2)(x+5)$, the remainder is $(x+3)$.

a Write the division identity $(P(x) = D(x)Q(x) + R(x)$ from this information.

Let $P(x) = (x-2)(x+5)Q(x) + (x+3)$

b Find the remainder when $P(x)$ is divided by $(x+5)$.

$P(-5) = -2$

14 **2012 HSC Extension 1 Band 4**

When the polynomial $P(x)$ is divided by $(x+1)(x-3)$, the remainder is $2x+7$.
What is the remainder when $P(x)$ is divided by $x-3$?

$P(x) = (x+1)(x-3)Q(x) + 2x + 7$

$P(3) = (x+1)(3-3)Q(x) + 2(3) + 7$

$= 13$

15 When a polynomial is divided by $(2x+1)(x-3)$, the remainder is $3x-1$.
What is the remainder when the polynomial is divided by $2x+1$?

Let $P(x) = (2x+1)(x-3)Q(x) + (3x-1)$

$P\left(-\frac{1}{2}\right) = 2.5$

16 **2014 HSC Extension 1 Band 4**

The remainder when $P(x) = x^4 - 8x^3 - 7x^2 + 3$ is divided by $x^2 + x$ is $ax + 3$.

What is the value of a?

$P(x) = x(x+1)Q(x) + ax + 3$

$P(-1) = (-1)^4 - 8(-1)^3 - 7(-1)^2 + 3$

$= 5$

$P(-1) = -1(-1+1)Q(-1) + a(-1) + 3$

$= -a + 3$

$\therefore 5 = -a + 3$

$a = -2$

17 A polynomial gives a remainder of 1 when divided by $(x\ +1)$, and a remainder of 5 when divided by $(x-3)$. What is the remainder when the same polynomial is divided by $(x+1)(x-3)$?

$P(x)=(x+1)(x-3)Q(x)+ax+b$

$P(-1)=1$

$P(3)=5$

$a=1, b=2$

$\therefore$ remainder is $x+2$

18 When x^5+3x^3+ax+b is divided by x^2-1, the remainder is $2x-7$. Find a and b.

$P(x)=(x-1)(x+1)Q(x)+2x-7$

$P(1)=1^5+3(1)^3+a(1)+b=2(1)-7$

$\therefore 4+a+b=-5$

$P(-1)=(-1)^5+3(-1)^3+a(-1)+b=2(-1)-7$

$\therefore -4-a+b=-9$

Solving simultaneously, we get

$a=-2, b=-7$

19 When a polynomial $P(x)$ is divided by x^2-5, the remainder is $x+4$.
Find the remainder when $P(x)+P(-x)$ is divided by x^2-5

$P(x)=(x^2-5)Q(x)+x+4$

$P(-x)=((-x)^2-5)Q(-x)+(-x)+4$

$=(x^2-5)Q(-x)-x+4$

$P(x)+P(-x)=(x^2-5)Q(x)+x+4+(x^2-5)Q(-x)-x+4$

$=(x^2-5)Q(x)+(x^2-5)Q(-x)+8$

$=(x^2-5)\big(Q(x)+Q(-x)\big)+8$

$\therefore$ remainder is 8

20 $P(x)$ is an odd polynomial of degree 3.
It has $x + 4$ as a factor, and when divided by $x - 3$ the remainder is 21. Find $P(x)$.

Hint: what terms can an odd polynomial have?

Odd polynomial must be in the form $ax^3 + bx$ (odd powers only for $P(-x) = P(x)$

By solving simultaneously $P(3) = 21$ and $P(-4) = 0$,

$a = -1, b = 16$

$\therefore P(x) = -x^3 + 16x$

21 Find p so that $(x - p)$ is a factor of $4x^3 - (10p - 1)x^2 + (6p^2 - 5)x + 6$.

$P(p) = 4p^3 - (10p - 1)p^2 + (6p^2 - 5)p + 6 = 0$

$p^2 - 5p + 6 = 0$

$(p - 2)(p - 3) = 0$

$\therefore p = 2, 3$

22 Find the real number a such that $(x - 1 - a)$ is a factor of $P(x) = x^3 - 3ax - 9$.

$P(1 + a) = (1 + a)^3 - 3a(1 + a) - 9 = 0$

$a^3 - 8 = 2$

$a = 2$

SOLVING POLYNOMIAL EQUATIONS

Integer Root Theorem
If all coefficients of $P(x)$ are integers, then all *integer roots* (if they exist) are factors of the constant term. $P(x) = x^3 + 2x^2 - 5x - 6 = (x+1)(x-2)(x+3)$, the integer roots are factors of -6 $P(x) = x^3 + x^2 + x + 6 = (x+2)(x^2 - x + 3)$ The integer root is a factor of 6

Example Find integer roots using factor theorem.

Find linear factors of $x^3 - 2x^2 - 2x - 3 = 0$

Test $\pm 1, \pm 3$

$P(1) = 6 \neq 0$ $\quad\therefore x - 1$ not a factor.

$P(-1) = -4 \neq 0$ $\quad\therefore x + 1$ not a factor.

$P(3) = 0$ $\quad\therefore x - 3$ is a factor.

$P(-3) = -4 \neq 0$ $\quad\therefore x + 3$ not a factor.

1 Use the factor theorem and trial and error to find **ONE** linear factor of these polynomials.

a $P(x) = x^3 + 2x^2 + 7x + 6$

$(x+1)$ is a factor

b $P(x) = x^3 + 2x^2 - x - 2$

$(x-2), (x-1), (x+1)$ are factors

c $P(x) = x^3 + x^2 + x + 6$

$(x+2)$ is a factor

2 Use the factor theorem to find **all three** linear factors of each of these polynomials.

a $P(x) = x^3 - 2x^2 - x + 2$

$(x-2)(x-1)(x+1)$

b $P(x) = x^3 - 2x^2 - 5x + 6$

$(x-1)(x+2)(x-3)$

c $P(x) = x^3 - 4x^2 + x + 6$

$(x+1)(x-2)(x-3)$

SOLVING POLYNOMIALS

- To solve a polynomial, we can factorise it and use the null factor law.
  1. Use integer root theorem to find an integer zero $x = a$ of $P(x)$.
  2. Then use polynomial long division to factor $P(x)$ in the form $P(x) = (x - a)Q(x)$
  3. Factorise $Q(x)$.
  4. Solve for x using null factor law.

Example Factorise a polynomial using factor theorem.

Solve $P(x) = x^3 + 2x^2 - 5x - 6 = 0$ Find integer root Test factors of -6: $P(1) = -8$, so $(x - 1)$ is not a factor. $P(-1) = 0$, so $(x + 1)$ is a factor. Long division $$\begin{array}{r} x^2 + x - 6 \\ x+1 \overline{) x^3 + 2x^2 - 5x - 6} \\ \underline{x^3 + x^2 } \\ x^2 - 5x - 6 \\ \underline{x^2 + x } \\ -6x - 6 \\ \underline{-6x - 6} \\ 0 \end{array}$$ $P(x) = (x + 1)(x^2 + x - 6)$ Factorise $P(x) = (x + 1)(x + 3)(x - 2)$ Solve $P(x) = (x + 1)(x + 3)(x - 2) = 0$ $\therefore x = -1, -3, 2$	How did we know to test factors of -6 to find a factor of the polynomial? *The integer root theorem says that if all the coefficients are ……………, the integer roots are all factors of the …………… ………………* Why must the remainder be zero when dividing this polynomial by $x + 1$? *$(x + 1)$ is a ……………, meaning it …………… evenly with no remainder.* How did we factorise $(x^2 + x - 6)$ to $(x + 3)(x - 2)$? *We factorised using the ………… method.* What law did we use the solve $$(x + 1)(x + 3)(x - 2) = 0$$ ……… ……………… law

Factorise these polynomials, and hence solve $P(x) = 0$.

a $P(x) = x^3 - 4x^2 + x + 6$

$x = -1, 2, 3$

b $P(x) = x^3 + 6x^2 - x - 30$

$x = -5, -3, 2$

c $P(x) = 2x^3 - 13x^2 + 13x + 10$

$x = -\frac{1}{2}, 2, 5$

d $P(x) = x^3 + 4x - 5$

$x = 1$ only real solution

1 $P(x) = x^3 - 3x - 2$

a Show that $(x + 1)$ is a factor of $P(x)$.

$P(-1) = (-1)^3 - 3(-1) - 2 = 0$

b Hence by using long division, write: $P(x) = (x + 1)(\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

$P(x) = (x + 1)(x^2 - x - 2)$

c By factorising the resultant quadratic, express $P(x)$ as a product of linear factors.

$P(x) = (x + 1)^2(x - 2)$

2 For each of the following cubic equations, follow these steps.

1. Use the factor theorem to find a factor.
2. Use long division to find the quotient.
3. Factorise the quotient.
4. Write the polynomial in a fully factorised form.
5. Solve for x using the null factor law.

a $x^3 - 4x^2 + x + 6 = 0$

$x = -1, 2, 3$

b $x^3 - 6x^2 + 11x - 6 = 0$

$x = 1,2,3$

c $x^3 - 3x^2 - 16x - 12 = 0$

$x = -2, -1, 6$

d $2x^3 - 3x^2 = 11x - 6$

$x = -2, 0.5, 3$

3 Solve by first taking out a common factor.

a $2x^3 - 14x^2 + 14x + 30 = 0$

$x = -1, 3, 5$

b $3x^3 + 12x^2 + 3x - 18 = 0$

$x = -3, -2, 1$

4 $P(x) = 2x^3 - 9x - 27$

a Show that $x - 3$ is a factor of $P(x)$

$P(3) = 2(3)^3 - 9(3) - 27 = 0$

b Hence by using long division, write: $P(x) = (x - 3)(\dots\dots\dots\dots\dots\dots\dots)$

$(x - 3)(2x^2 + 6x + 9)$

c By using the discriminant, show that 3 is the only real zero of this polynomial.

Δ of $2x^2 + 6x + 9$ is $\Delta = 6^2 - 4(2)(9) < 0$

So $x = 3$ is the only zero.

5 Explain why $(x - 2)(x^2 - 3x + 3) = 0$ has only one real solution.

Δ of $x^2 - 3x + 3$ is $\Delta = (-3)^2 - 4(1)(3) < 0$

6 Explain why $x^4 + x^2 = 0$ has only one real solution.

$x^4 + x^2 = x^2(x^2 + 1)$

$x^2 = 0, \ \ x = 0$

$x^2 + 1 = 0, \ \ x^2 = -1$

No real solution for $x^2 = -1$, so only real solution is $x = 0$

7 Show that $P(x) = x^3 + x + 3$ does not have an integer zero by testing all the possible integer zeros.

Test $P(-3), P(-1), P(1), P(3)$. None give zero.

8 Use the quadratic formula to solve for x, expressing your answers in exact form.

a $(x-1)(x^2-2x-4)=0$

$x = 1, 1 \pm \sqrt{5}$

b $(x+2)(x^2+6x+10)=0$

$x = -2$

9 Solve for x without long division by factorising first.

a $x^3 - x^2 = 0$

$x = 0, 1$

b $x^3 - x^2 - 12x = 0$

$x = -3, 0, 4$

c $2x^5 + 4x^4 + 2x^3 = 0$

$x = -1, 0$

10

a Show that $P(x) = x^3 - 8x^2 + 9x + 18$ is divisible by both $(x-3)$ and $(x+1)$.

$P(3) = 0$

$P(-1) = 0$

b By considering the leading term and constant term, express $P(x)$ as a product of three linear factors and hence solve $P(x) = 0$

$(x-3)(x+1)(x-6)$

$x = 3, -1, 6$

11

a Show that $P(x) = 2x^3 - x^2 - 13x - 6$ is divisible by both $(x-3)$ and $(2x+1)$.

$P(3) = 0$

$P\left(-\frac{1}{2}\right) = 0$

b By considering the leading term and constant term, express $P(x)$ as a product of three linear factors and hence solve $P(x) = 0$

$(2x+1)(x-3)(x+2)$

$x = -2, -\frac{1}{2}, 3$

12 A fast way of factorising polynomials is by equating coefficients.

Suppose you know that $3x^3 - 8x^2 - x + 10 = (x+1)(ax^2 + bx + c)$

Step 1: Compare leading coefficients to find a:

The coefficient of x^3 on the LHS is ……

3

The coefficient of x^3 on the RHS would be ………..

a

Therefore, $a =$ ……

3

Step 2: Compare constant terms to find c:

The constant term on the LHS is ……

10

The constant term on the RHS would be ………..

c

Therefore, $c =$ ……

10

Step 3: Rewrite with values of a and c:

$$3x^3 - 8x^2 - x + 10 = (x+1)(3x^2 + bx + 10)$$

Step 4: Equate other coefficients to find b:

The coefficient of x on the LHS is ……

-1

The coefficient of x on the RHS would be …………………

$b + 10$

Therefore, $b =$ ……

-11

Step 5: Rewrite and factorise.

$$\begin{aligned} 3x^3 - 8x^2 - x + 10 &= (3x^2 - 11x + 10)(x+1) \\ &= (3x-5)(x-2)(x+1) \end{aligned}$$

13 Factorise by equating coefficients, given $x^3 + x^2 - 4x - 4 = (x + 1)(ax^2 + bx + c)$

$(x + 1)(x - 2)(x + 2)$

14 Show that $(2x - 1)$ is a factor of $P(x) = 6x^3 + 23x^2 - 33x + 10$.
Then fully factorise $P(x)$ (by long division or comparing coefficients) and state the zeros.

$(2x - 1)(3x - 2)(x + 5)$

15 $P(x) = x^3 - 6x + 4$

- **a** Find a factor of $P(x)$ using the factor theorem by testing the factors of 4.
- **b** Express $P(x)$ as the product of a linear and quadratic factor.
- **c** Hence solve $P(x) = 0$.

$x = 2, -1 \pm \sqrt{3}$

16 $P(x) = 2x^3 + x^2 - 13x + 6$. Evaluate $P(0.5)$ and hence find the zeroes of $P(x)$.

$x = \frac{1}{2}, -3, 2$

17 $P(x) = 6x^3 + x^2 - 5x - 2$. Evaluate $P\left(-\frac{2}{3}\right)$ and hence find the zeroes of $P(x)$.

$x = -\frac{2}{3}, -\frac{1}{2}, 1$

18 Use trial and error to find as many integer zeroes of $P(x)$ as possible, then use long division to find the other factors.

a $P(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

$(x-1)(x+1)(x-3)(2x+1)$

b $P(x) = 6x^4 - 25x^3 + 17x^2 + 28x - 20$

$(2x-5)(3x-2)(x+1)(x-2)$

19 Suppose $6x^3 + ax^2 - 11x + 15 = (2x-5)(px^2 + qx + r)$.
Find a, p, q, r by equating coefficients.

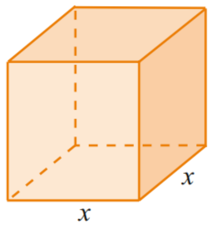
$p = 3, r = -3, q = 1, a = -13$

20 A box is to be built with a square base and height 2 cm more than the length of the base.

a Write an expression for the volume of the box.

b If the volume is to be 45 cm^3, write a polynomial equation modelling this scenario.

c Find the dimensions that will give this volume.



a) $x^2(x+2)$

b) $x^3+2x^2-45=0$

c) $x=3$

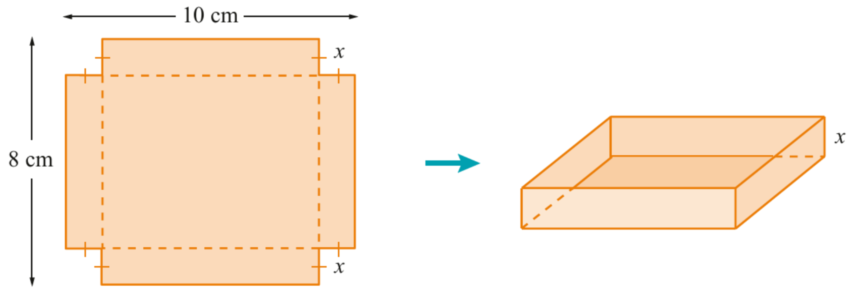
∴ 3 cm x 3 cm x 5 cm

21 Four identical squares are cut from the corners of a rectangular sheet of metal 8 cm x 10 cm. The sides are folded up along the dotted lines to form a rectangular box, as shown.

a Find an expression for the volume of the box.

b Find the side length of the square that should be cut from each corner if the volume is to be 48 cm^3.

c Hence state the dimensions of the rectangular box with this volume.



a) $x(8-2x)(10-2x)$

b) $4x^3-36x^2+80x-48=0$

c) $x=1$ or 2

∴ 8 x 6 x 1 or 6 x 4 x 2 cm

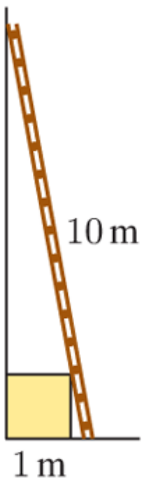
22 Factorise $x^4 - x^3 - 5x^2 + 12$.

$(x-2)^2(x^2+3x+3)$

23 A ladder of length 10 metres is leaning against a wall so that it is just touching a cube of edge length one metre.

a Let $x + 1$ be the height up the wall that the ladder reaches and $y + 1$ be the distance along the ground. Show that this scenario can be modelled using

$$x^4 + 2x^3 - 98x^2 + 2x + 1 = 0$$



Hint: Use Pythagoras and similar triangles.

Using Pythagoras' theorem:

$(x+1)^2 + (1+y)^2 = 10^2$

Using similar triangles, $\frac{y}{1} = \frac{1}{x}$

$\therefore y = \frac{1}{x}$

$(x+1)^2 + \left(1 + \frac{1}{x}\right)^2 = 10^2$

Expand, multiply every term by x^2, then rearrange.

$x^4 + 2x^3 - 98x^2 + 2x + 1 = 0$

b Use an DESMOS to solve $x^4 + 2x^3 - 98x^2 + 2x + 1 = 0$ and hence find the height up the wall the ladder reaches to 2 decimal places.

$x \approx 0.11$ m or 8.94 m

So, height up the wall is 1.11 m or 9.94 m