Student Name

Mathematics Stage 5 Paths

PROPERTIES OF GEOMETRICAL FIGURES

Book 5 Apply logical reasoning to numerical problems involving plane shapes

Version: 260207

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OVERVIEW

SYLLABUS CONTENT

MA5-GEO-P-01 establishes conditions for congruent triangles and similar triangles and solves problems relating to properties of similar figures and plane shapes

Apply logical reasoning to numerical problems involving plane shapes

- Apply geometrical facts, properties and relationships to find the sizes of unknown sides and angles of plane shapes in diagrams, providing appropriate reasons
- Define the exterior angle of a convex polygon
- Establish the sum of the exterior angles of any convex polygon is $360°$ and verify this result
- Apply the result for the sum of the interior angles of a triangle to find, by dissection, the sum of the interior angles of a polygon with more than 3 sides
- Apply the results for the interior and exterior angles of polygons to solve problems involving polygons

REVIEW OF PRIOR KNOWLEDGE

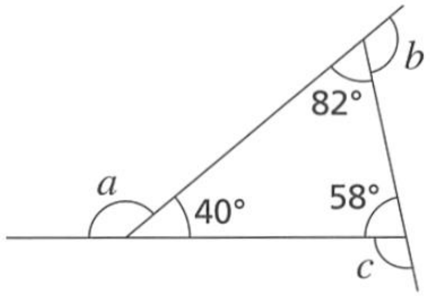
1 Evaluate $(n-2)\times 180°$ if:

a $n=6$

b $n=10$

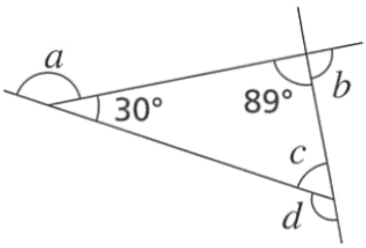
c $n=22$

2 Find the value of each pronumeral in this diagram.

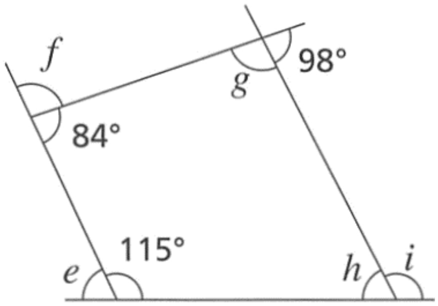


3 Work out the value of the missing angles.

a



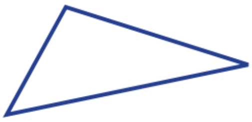
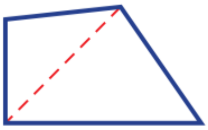
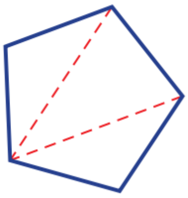
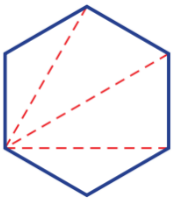
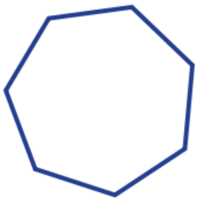
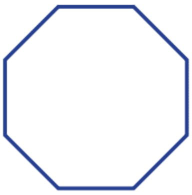
b



ANGLE SUM OF A POLYGON

- A **polygon** is a 2-dimensional shape with at least three straight sides and angles.
- The number of interior angles is equal to the number of sides.

Investigation Angle sum of a Polygon.

Polygon	Sides	Triangles	Angle Sum
Triangle 	3	1	$1 \times 180° = 180°$
Quadrilateral 	2	2	$\dots\dots\dots \times 180° = \dots\dots\dots$
Pentagon 			
Hexagon 			
Heptagon 			
Octagon 			
n-gon			$\dots\dots\dots \times 180°$

Interior Angle Sum of a Polygon

The interior angle sum of a polygon with n sides is:

$$S = 180(n - 2)$$

Example Use the formula for the angle sum of a polygon.

Find the angle sum of a heptagon.	How did we know to let $n = 7$ in the formula?
$S = 180(n - 2)$ $= 180(7 - 2)$ $= 180(5)$ $= 900°$	*A heptagon has …… sides.*

Find the angle sum of a:

a Hexagon

720°

b Decagon (10 sides)

1440°

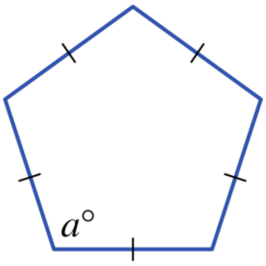
Key ideas

1. The interior angle sum of any polygon is given by the formula ………………………

1 Find the interior angle sum of these polygons.

a Octagon (8 sides) 1080°	**b** nonagon (9 sides) 1260°	**c** heptagon (7 sides) 900°
d 11-sided polygon 1620°	**e** 15-sided polygon 2340°	**f** 45-sided polygon 7740°

2 Regular polygons have equal interior angles.



a What is the angle sum of a pentagon?

540°

b What is the size of each interior angle of a pentagon?

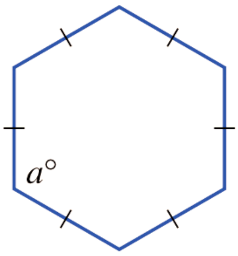
108°

3 Find the size of an interior angle for these regular polygons with the given angle sum.

a regular octagon (1080°) 135°	**b** regular nonagon (1260°) 140°	**c** regular decagon (1440°) 144°

4 Find the size of each interior angle of these regular polygons.
Round the answer to one decimal place where necessary.

a regular hexagon



120°

b regular heptagon

128.6°

c regular 32-sided polygon

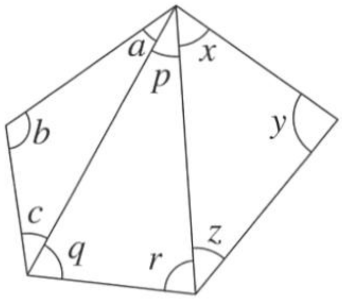
168.8°

5 Use the diagram and angle sum of a triangle to explain why the sum of the interior angles of a pentagon is 540°.

$a + b + c =$ ………° (………………………………) 180° (∠ sum of Δ)

$p + q + r =$ ………° (………………………………) 180° (∠ sum of Δ)

$x + y + z =$ ………° (………………………………) 180° (∠ sum of Δ)



$\therefore (a + b + c) + (p + q + r) + (x + y + z) =$ ………° 540°

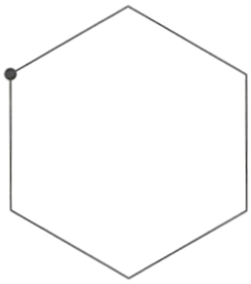
Therefore, the interior angle sum of a pentagon is ………° 540°

6

a Split the hexagon into triangles by drawing diagonals from the indicated vertex.

b How many triangles are formed inside the hexagon? 4

c Work out the sum of the interior angles of a hexagon. 720°



7 Find the number of sides of a polygon with the given angle sums.

a 1260° $S = 180(n - 2)$ $1260 = 180(n - 2)$ 9	**b** 2340° 15	**c** 3420° 21
d 720° 6	**e** 3240° 20	**f** 1260° 9

8 The angle sum of a regular polygon is 2520°

a How many sides does it have? 16

b Find the size of each angle. 157.5°

9 Let S be the angle sum of a regular polygon with n sides.

a Write the rule for the size of an interior angle in terms of S and n.

Each interior angle = S ÷ n

b Write a rule for the size of an interior angle in terms of n only.

Substitute S = 180(n - 2):

Each interior angle = $180(n - 2) \div n$

Or simplified: $180 - \frac{360}{n}$

c Use your rule to find the size of an interior angle of these polygons.
Round to two decimal places where appropriate.

i regular dodecagon ………………… **ii** 82-sided regular polygon …………………

150° 175.61° (to 2 d.p.)

10 Given that each interior angle of a regular polygon is $I = \frac{180(n-2)}{n}$, rearrange the formula so that n is the subject.

$n = -\frac{360}{(I - 180)}$

11 Find the number of sides of a regular polygon if each interior angle is:

a $150°$ 12	**b** $175°$ 72	**c** $162°$ 20
d $140°$ 9	**e** $165.6°$ 25	**f** 147.272727…° 11

12 Consider a regular polygon with a very large number of sides (n).

a What shape does this polygon look like?

circle

b Is there a limit to the size of a polygon angle sum as n increases?

Using S = 180(n - 2), as n gets larger, S keeps growing without bound.

c What size does each interior angle approach as n increases?

Each interior angle = $180(n - 2) \div n = 180 - \frac{360}{n}$

As n increases, $\frac{360}{n}$ approaches 0, so each interior angle approaches 180°.

13 Is it possible for a regular polygon to have interior angles of size $132°$?
Explain why or why not.

Using the formula: $n = \frac{360}{(180-I)}$

For $I = 132°, n = 7.5$

∴ no

PROBLEMS INVOLVING ANGLE SUM OF A POLYGON

- If a problem involves finding a missing angle inside a polygon:
  1. Determine the angle sum using $S = 180(n - 2)$.
  2. Form an equation to find the missing angle.

Example Solve problems involving angle sum of a polygon.

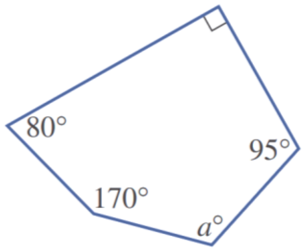
Find the value of a in this pentagon.

1. Angle sum

$S = 180(n - 2)$

$= 180(5 - 2)$

$= 540°$



2. Form an equation

$a + 170 + 80 + 90 + 95 = 540$

$a + 435 = 540$

$a = 105$

How did we know $n = 5$?

The figure has ………… sides

Describe the process of solving the equation

$a + 170 + 80 + 90 + 95 = 540$.

First, we ……………… the LHS,

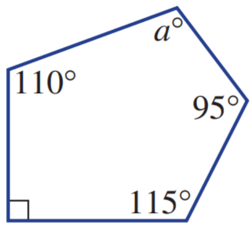
Then we …………… ……… from both sides.

Find the missing angle.

a

1. Angle sum

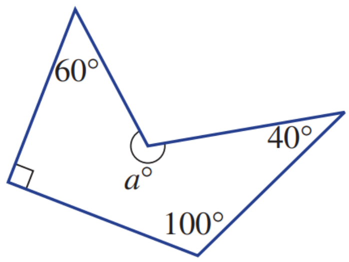
$S = 180(n - 2)$



2. Form an equation

130°

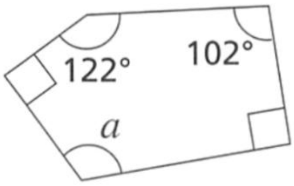
b



250°

1 Find the value of a in these polygons. Remember to find the angle sum first.

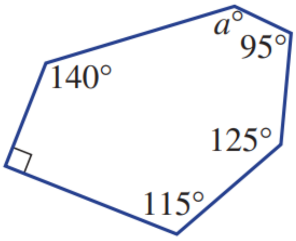
a



$S = 180(n-2)$

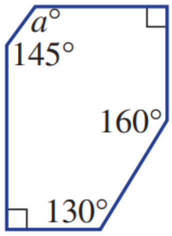
136°

b



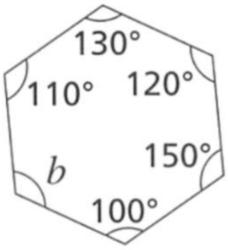
155

c



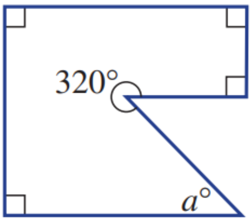
105

d



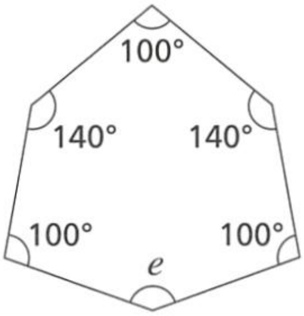
110°

e



40

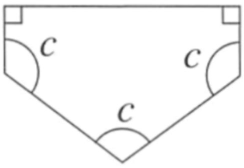
f



140°

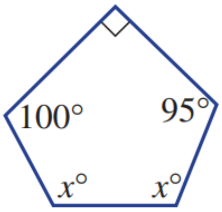
2 Find the value of the pronumeral in these diagrams.

a



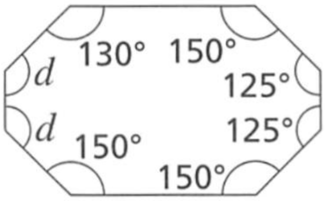
120°

b



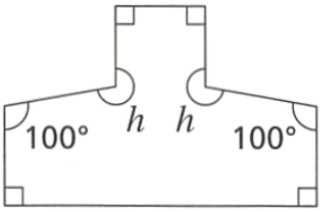
127.5

c



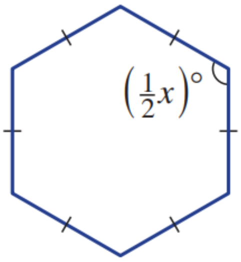
125°

d



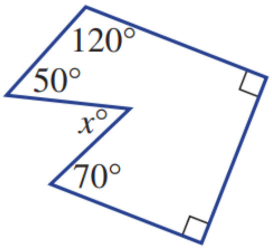
260°

e



240

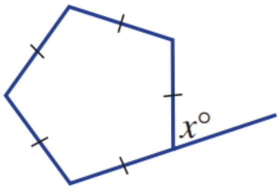
f



60

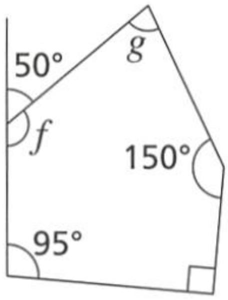
3 Find the value of the pronumeral in these diagrams.

a



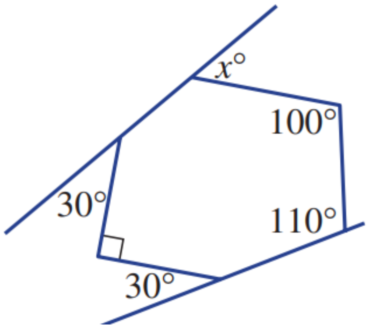
72

b



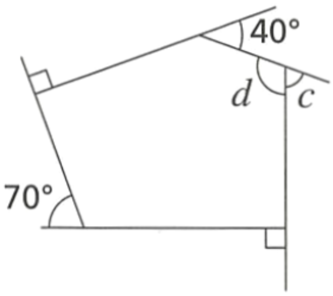
$f = 130°, g = 75°$

c



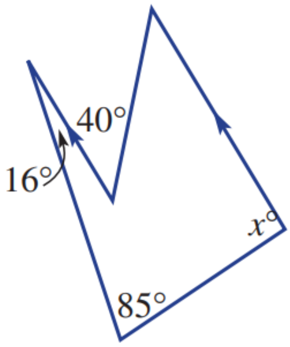
60

d



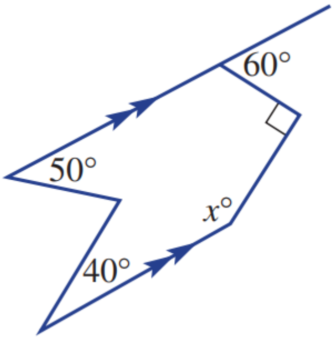
$c = 70°, d = 110°$

e



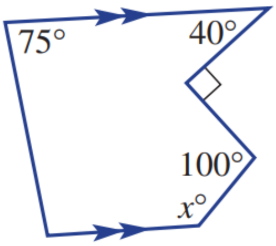
79

f



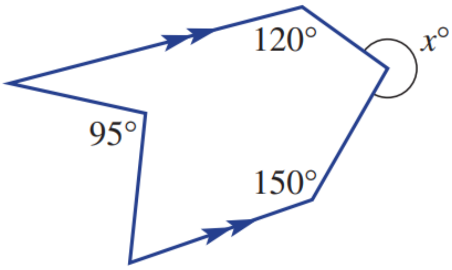
150

g



130

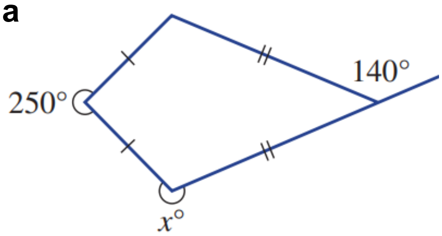
h



270

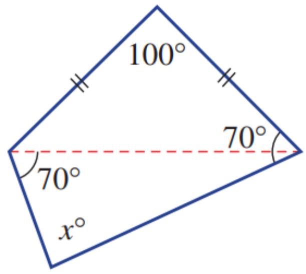
4 Find the value of x in these diagrams.

a



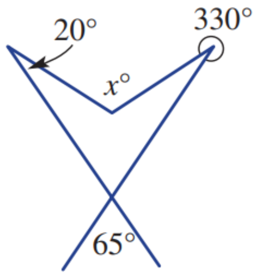
255

b



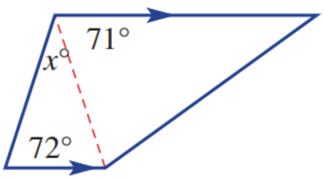
80

c



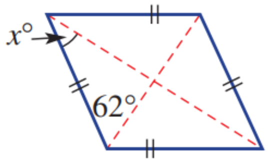
115

d



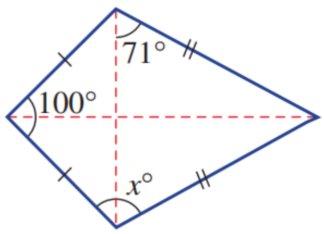
37

e



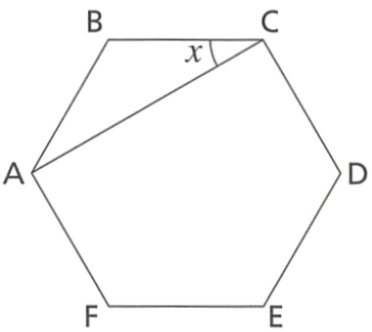
28

f



111

6 The diagram shows a regular hexagon ABCDEF. The line AC is constructed.
Work out the size of angle x.



ΔABC is isosceles (two equal sides)

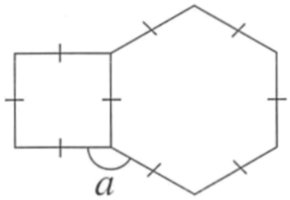
$\angle B = 120°$ (interior angle of a hexagon)

$x + x + 120 = 180$ (angle sum of triangle)

$\therefore x = 30°$

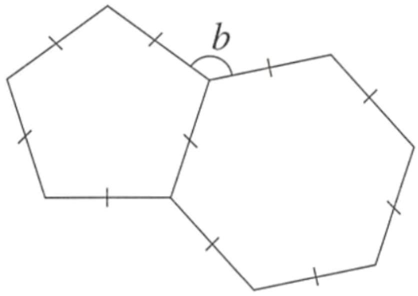
7 Work out the value of the pronumeral, given that these are regular polygons.

a



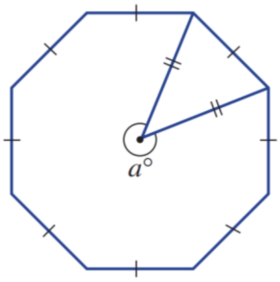
150°

b



132°

8 Work out the value of a

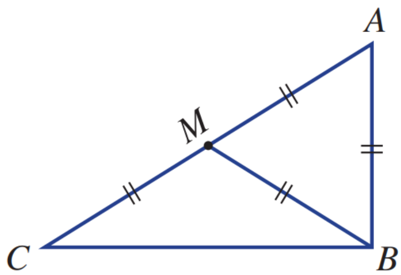


Let θ be the angle inside the triangle at the centre of the octagon.

$\theta = 360 \div 8 = 45°$

$x = 360 - 45° = 315°$

9 Prove that $\angle ABC = 90°$



∠AMB = 60° (△ ABM is equilateral).

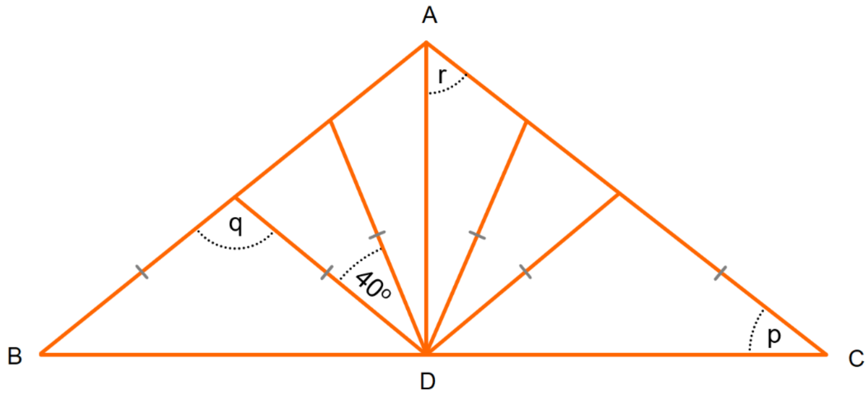
∠BMC = 120° (supplementary).

Therefore, ∠MBC = 30° (△ MBC is isosceles).

So ∠ABC = ∠ABM + ∠MBC = 60°+ 30°= 90°

10 A framework for a symmetrical roof is shown. AD is perpendicular to BC.

Find the value of the angles marked p, q, r.

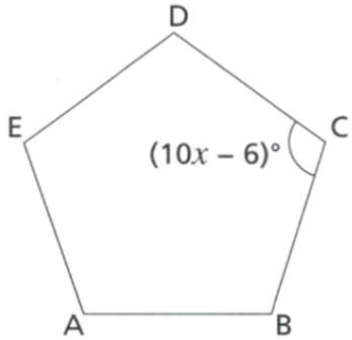


$q = 110°$

$p = 35°$

$r = 55°$

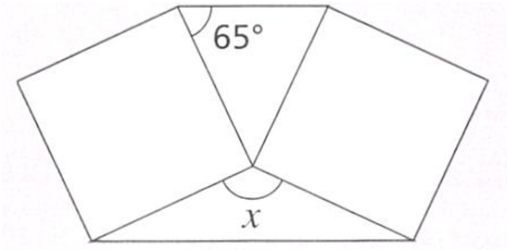
11 ABCDE is a regular pentagon. Work out the value of x.



$5(10x-6)=540$ (angle sum of pentagon)

$x=11.4°$

12 The diagram shows two identical squares. Find the size of angle x.



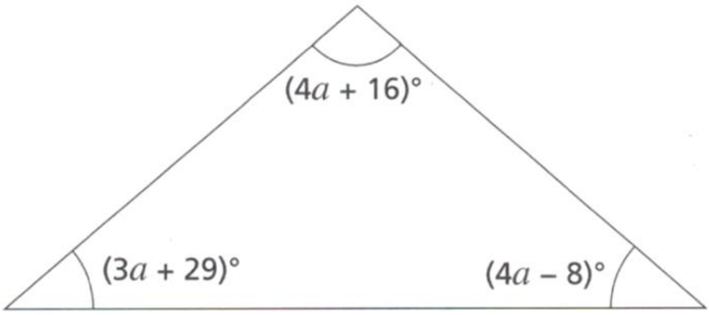
Let θ be the angle opposite x.

Angle sum at a point: $x+90+90+\theta=360°$

$\theta=180-2(65)=50°$ (angles in isosceles triangle)

$\therefore x=130°$

13 Show that this triangle is isosceles.



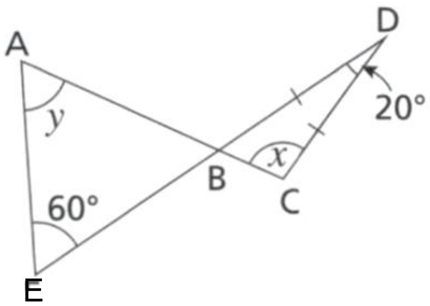
Sum of all angles: 11a + 37 = 180°

a = 13°

Sub into the expressions for the angles gives 44°, 68°, 68°.

Since two angles are equal, it must be an isosceles triangle.

14 Show that angle y is half the size of angle x.



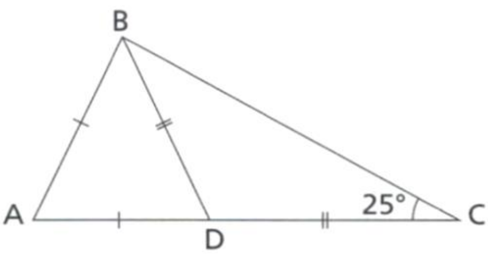
∴ ∠DBC = 80° (angles in a triangle add up to 180°; two angles in an isosceles triangle are equal)

∠ABE = 80° (vertically opposite angles are equal)

∠BAE = 40° (angles in a triangle add up to 180°)

40° is half of 80°, as required

15 Show that $\angle BAD = 80°$. Give a reason for each stage of your working.



∠DBC = 25° (two angles in an isosceles triangle are equal)

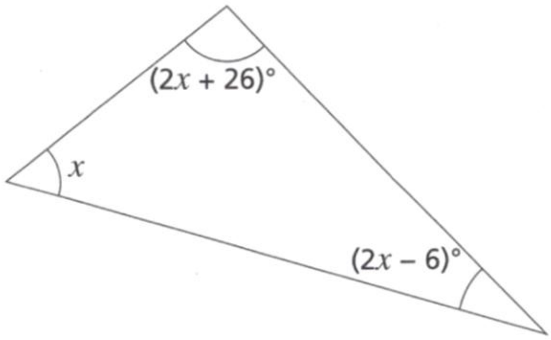
∠BDC = 130° (angles in a triangle add up to 180°)

∠ADB = 50° (angles on a straight line add up to 180°)

∠DBA = 50° (two angles in an isosceles triangle are equal)

∠BAD = 80° (angles in a triangle add up to 180°)

16 Show that this shape is a right-angled triangle.

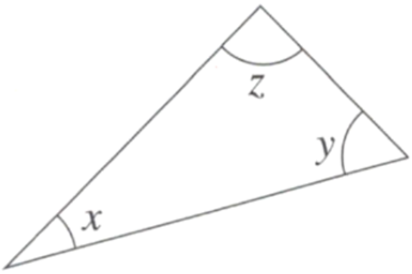


Angles in a triangle add up to 180°

$5x + 20° = 180°, x = 32°$

The angles are then 32°, 58° and 90°

17 Consider the triangle.



a If $x + y = z$, show that z must equal $90°$

$x + y + z = 180°$ (angle sum of triangle)

$\therefore z + z = 180°$

$\therefore z = 90°$

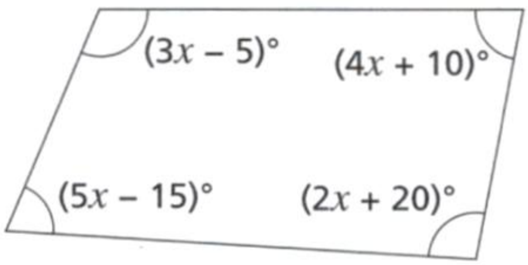
b If $x + y = 2z$, show that z must be equal to $60°$

$x + y + z = 180°$ (angle sum of triangle)

$2z + z = 180°$

$z = 60°$

18 Show that this shape is a parallelogram. Clearly justify your answer.



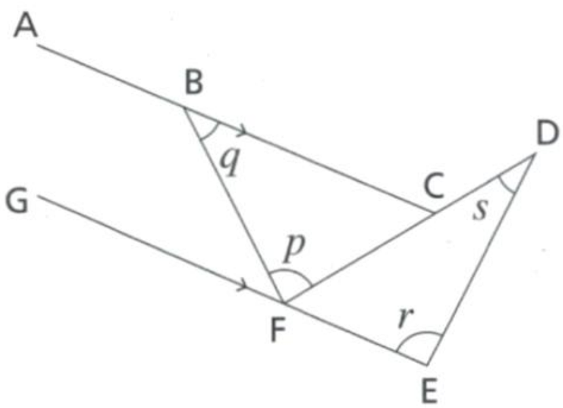
Angle sum of quadrilateral: $14x + 10 = 360°$

$x = 25°$

Substituting into expressions for angles, 70, 110, 70, 110

Co-interior angles sum to 180°, so opposite sides are parallel.

19 Show that $p + q = r + s$



$\angle BFG = q$ (alternate angles are equal)

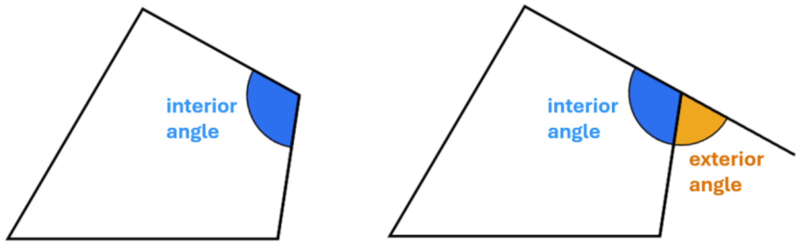
∠DFE = $180° - (r + s)$ (angles in a triangle add up to 180°)

On a straight line at F: $q + p + 180 - (r + s) = 180°$;

Hence $q + p = r + s$

EXTERIOR ANGLE SUM OF A POLYGON

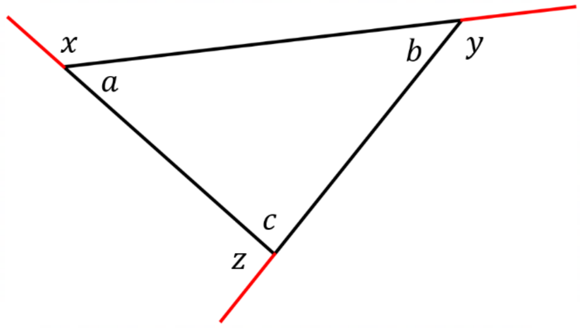
- When we extend one side of a polygon, it forms an **exterior angle**.



Sum of Interior and Exterior Angles
The sum of an interior angle and its exterior angle is $180°$ (supplementary)

Investigation Sum of exterior angles of a polygon.

Consider this triangle.



$a + x = ...$

$b + y = ...$

$c + z = ...$

The sum of *all* the interior and exterior angles is:

$(a + x) + (b + y) + (c + z) = 3 \times 180$

$= ....$

The sum of interior angles is:

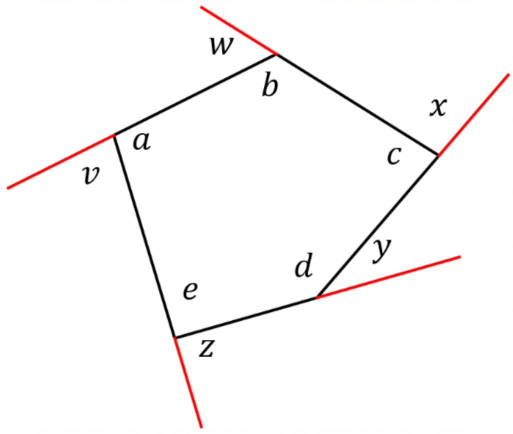
$a + b + c = ...$

So, the sum of exterior angles is:

$x + y + z = 540 - ....$

$= ....$

Consider this pentagon.



$a + v = ...$

$b + w = ...$

The sum of all the interior and exterior angles is:

$(a + v) + (b + w) + (c + x) + +y) + (e + z) = ...$

The sum of interior angles is:

$a + b + c + d + e = ...$

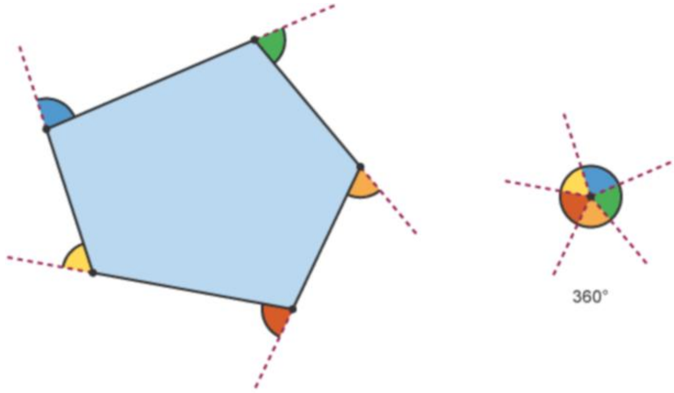
So, the sum of exterior angles is:

$v + w + x + y + z = ...$

Exterior angle sum https://www.geogebra.org/m/jgpwcukv
https://www.geogebra.org/m/fnkpeucj

Exterior Angle Sum of a Polygon

The exterior angle sum of any polygon is $360°$



Exterior Angle of a Regular Polygon

Exterior angle of a *regular* polygon with n sides:

$$E = \frac{360°}{n}$$

Example Find exterior angle of a regular polygon.

Find the size of each exterior angle of a regular decagon. $E = \frac{360°}{10} = 36°$ Each exterior angle of a regular polygon is $12°$ How many sides does it have? $12° = \frac{360°}{n}$ $n = \frac{360°}{12°} = 30$	If every polygon has an exterior angle sum of $360°$, why does this formula only work for a regular polygon? *A regular polygon has every .... the same size.*

a Find the size of each exterior angle of a regular octagon.

45°

b Find the size of each exterior angle of a regular 20-sided polygon.

18°

c Each exterior angle of a regular polygon is 15°. Find the number of sides of the polygon.

24 sides

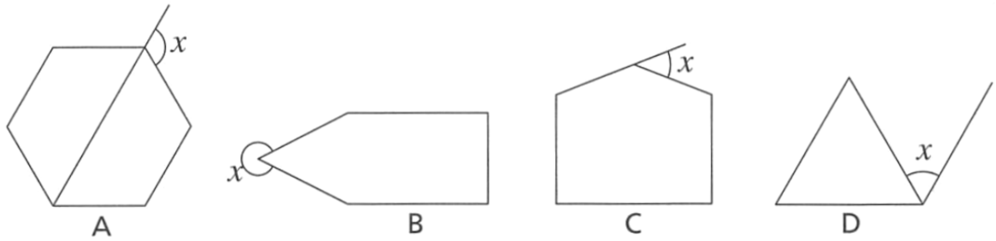
d Find the number of sides in a regular polygon with each exterior angle $10°$

36 sides

Key ideas

1. The exterior angle sum of any polygon is ....

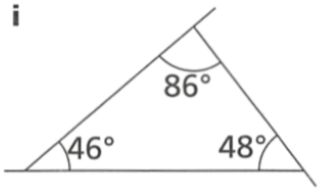
1 In which diagram is x an exterior angle?



C

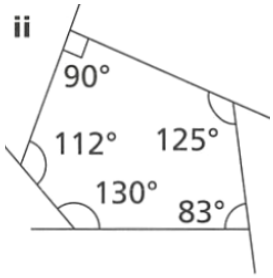
2 Work out the sizes of the exterior angles of each of these shapes.

i



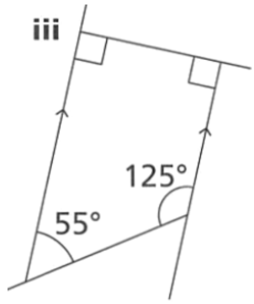
94°,132°, 134°

ii



90°, 55°, 97°, 50°, 68°

iii



90°, 90°, 55°, 125°

What is the sum of the exterior angles for each of the shapes above?

360°

3 Find the size of each exterior angle in a regular:

a Pentagon

72°

b Dodecagon

30°

c 18-sided polygon

20°

4

a Each exterior angle of a regular polygon is 15°. Find the number of sides of the polygon.

24 sides

b Find the number of sides in a regular polygon with each exterior angle of 18°

20 sides

5

a Explain why each exterior angle of a regular hexagon is $60°$

In a regular hexagon, all exterior angles are equal, so each exterior angle = 360° ÷ 6 = 60°.

b Work out the size of each exterior angle of a:

i Regular pentagon

72°

ii Regular heptagon

51.43°

iii Regular 30-sided polygon.

12°

c What happens to the size of each exterior angle as the number of sides of a regular polygon increases?

The exterior angle approaches 0°.

6 Find the number of sides of a regular polygon if each exterior angle is:

a $24°$

15

b $36°$

10

c $5°$

72

7 A 50-cent piece has 12 equal sides and 12 equal angles. Calculate the size of each:

a exterior angle

30°

b interior angle

150°

8

a Find the size of each exterior angle of a regular octagon.

45°

b Use the exterior angle to find the size of the interior angle. Give a reason for the answer.

135°. The interior and exterior angles at each vertex are supplementary (they sum to 180°), so I = 180° - E.

c Hence, find the angle sum of an octagon.

$S = (8 - 2) \times 180° = 6 \times 180° = 1080°$

9

a Find the size of each exterior angle of a regular 14-sided polygon.

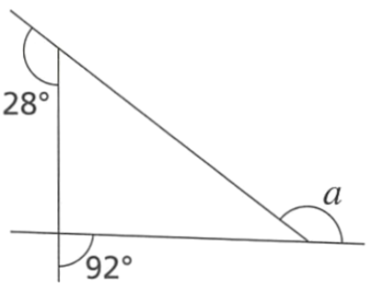
E = 360° ÷ 14 = 25.71°

b Find the size of each interior angle.

I = (14 - 2) × 180° ÷ 14 = 154.29°

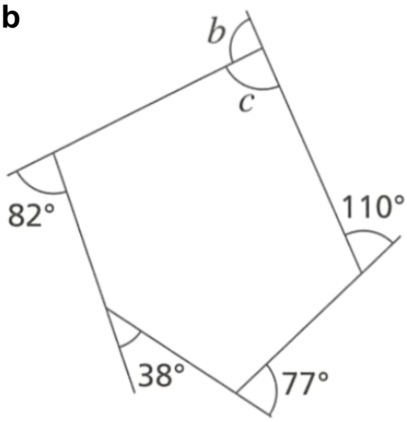
10 Determine the sizes of the labelled angles in these diagrams.

a



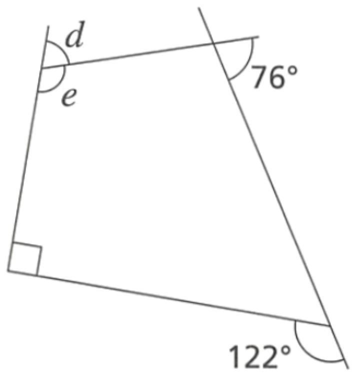
$a = 140°$

b



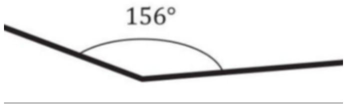
$b = 53°, c = 127°$

c



$d = 72°, e = 108°$

11 A regular polygon has a single interior angle as shown.
Work out the number of sides of the regular polygon

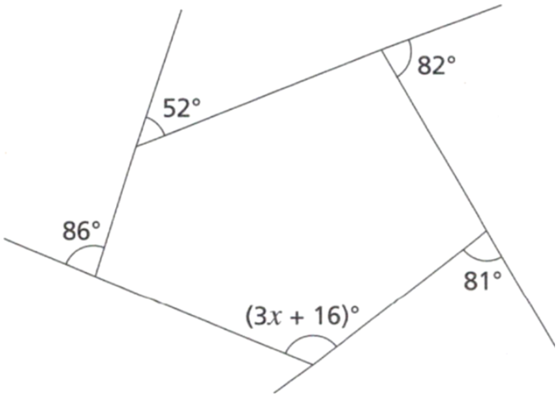


Exterior angle is 24°

$n = \frac{360}{24} = 15$

15 sides

12 Work out the value of x



Exterior angle of $(3x + 16)$ is 59°

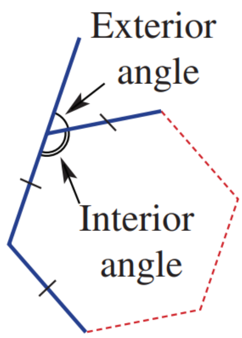
$3x + 16 + 59 = 180$

$x = 35°$

13 For a regular polygon with n sides:

a write the rule for the sum of interior angles (S)

$S = 180(n - 2)$



b write the rule for the size of each interior angle (I)

$I = \frac{180(n-2)}{n}$

c write the rule for the size of each exterior angle (E)

$E = \frac{360}{n}$

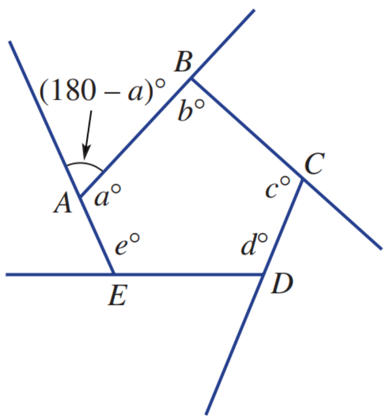
d use your rule from part **c** to find the size of the exterior angle of a regular decagon.

$E = \frac{360°}{10} = 36°$

14 If you follow the path around this pentagon clockwise starting and finishing at point A, you will have turned a total of 360°. Complete this proof of the angle sum of a pentagon (540°).

$(180 - a) + (180 - b) + \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

$(180 - c) \quad + \quad (180 - d) \quad + \quad (180 - e) \quad = \quad 360$



(sum of exterior angles is ....)

360°

$180 + 180 + \ldots\ldots + \ldots\ldots + \ldots\ldots - (a + b + \ldots\ldots + \ldots\ldots) = 360°$

180 180 180

$\ldots\ldots\ldots\ldots\ldots\ldots - (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) = 360°$

900 – $a + b + c + d + e$

$(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots°$

$a + b + c + d + e$ 540°

15 An irregular polygon has one exterior angle of 100° and all others are 13°.

a Explain why the sum of all the $13°$ angles is $260°$

360° - 100° = 260°

b Therefore, how many 13-degree angles are there?

260° ÷ 13° = 20 angles of 13°

c How many sides are there, in total? (hint: it's not 20)

21 sides in total.

d To find the number of sides, we can use the equation $100 + 13(n - 1) = 360°$, where n is the number of sides in the polygon. Explain why.

There's one 100° angle and $(n - 1)$ angles of 13°, where n is the total number of sides.

e Find the number of sides of the polygon using the equation.

$100 + 13(n - 1) = 360$

$13(n - 1) = 260$

$n - 1 = 20$

$n = 21$ sides

16 Find the number of sides in an irregular polygon with one exterior angle of 60° and all others 10°

$60 + 10(n - 1) = 360$

$10(n - 1) = 300$

$n - 1 = 30$

$n = 31$ sides

17 Find the number of sides in an irregular polygon with one exterior angle of 45° and all others 15°

$45 + 15(n - 1) = 360$

$15(n - 1) = 315$

$n - 1 = 21$

$n = 22$ sides

18 An irregular polygon has two exterior angles twice the size of the others.
If the other exterior angles are 15°, find the number of sides.

$2(30) + 15(n - 2) = 360$

$60 + 15(n - 2) = 360$

$15(n - 2) = 300$

$n - 2 = 20$

$n = 22$ sides

19 An irregular polygon has two exterior angles twice the size of the others.
If the other exterior angles are 20°, find the number of sides.

$2(40) + 20(n - 2) = 360$

$80 + 20(n - 2) = 360$

$20(n - 2) = 280$

$n - 2 = 14$

$n = 16$ sides

20 Four of the exterior angles of a pentagon are 90°, 60°, 70° and 90°
Work out the interior angles of the pentagon.

Four exterior angles are 90°, 60°, 70° and 90°.

Sum of these four: 90 + 60 + 70 + 90 = 310°

The fifth exterior angle: 360° - 310° = 50°

Each interior angle is supplementary to its exterior angle:

Interior angles: 90°, 120°, 110°, 90°, 130°

21 Is it possible for the exterior angle of a regular polygon to be 150°? Why or why not?

If each exterior angle is 150°:

$150 = \frac{360}{n}$

$n = \frac{360}{150} = 2.4$

Since a polygon must have a whole number of sides, an exterior angle of 150° is impossible for a regular polygon.

22 A regular polygon has exterior angles and interior angles in the ratio 2:7.

How many sides does it have?

E : I = 2 : 7

$\frac{E}{I} = \frac{2}{7}$

$\therefore I = 3.5E$

$E + I = 180$

$E + 3.5E = 180$

$\therefore E = 40°$

Using $E = \frac{360°}{n}$

$n = \frac{360}{40} = 9$ sides